Optimal Public Debt with Life Cycle Motives*

William B. Peterman
Federal Reserve Board

Erick Sager
Bureau of Labor Statistics

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Abstract

Public debt can be optimal in standard incomplete market models with infinitely lived agents, since capital crowd-out induces a higher interest rate that encourages agents to hold more precautionary savings against uninsurable idiosyncratic labor risk. Optimal policy of this class of economies depends on household savings behavior, yet abstracts from empirical life cycle savings patterns. This paper incorporates a life cycle into the incomplete markets model in order to generate realistic savings patterns: agents enter the economy with little wealth and accumulate savings over their lifetimes. This paper finds that while public debt equal to 24% of output is optimal in the infinitely lived agent model, public savings equal to 59% of output is optimal in the life cycle model. Even though a higher level of public debt similarly encourages life cycle agents to hold more savings during their lifetimes, the act of accumulating this savings mitigates the welfare benefit from public debt. Moreover, not accounting for the life cycle when computing optimal policy reduces average welfare by at least 0.5% of expected lifetime consumption.

Keywords: Government Debt; Life Cycle; Heterogeneous Agents; Incomplete Markets
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1 Introduction

Motivated by the prevalence of government borrowing across advanced economies, previous work demonstrates that government debt can be optimal in a standard incomplete markets model with infinitely lived agents. For example, in their seminal work, Aiyagari and McGrattan (1998) find that a large quantity of public debt is optimal when such an economic environment is calibrated to the U.S. economy. Public debt is optimal because it crowds out the stock of productive capital and leads to a higher interest rate that encourages households to save more. As a result, households are better self-insured against idiosyncratic labor earnings risk and, therefore, are less likely to be liquidity constrained. While household savings behavior is central to public debt being optimal, previous work largely examines optimal policy in economies inhabited by infinitely lived agents. Such economic environments abstract from empirically relevant life cycle characteristics that influence savings decisions and that can, therefore, affect optimal debt policy.

This paper characterizes the effect of a life cycle on optimal public debt and inspects the mechanisms through which a life cycle affects optimal policy. In order to determine the effect of the life cycle, we contrast optimal policy in two model economies: (i) the standard incomplete markets model with infinitely lived agents, and (ii) a life cycle model. We find that the optimal policies are strikingly different between the two models. In the infinitely lived agent model, it is optimal for the government to be a net borrower with public debt equal to 24 percent of output. In contrast, in the life cycle model, we find that it is optimal for the government to be a net saver, not a net borrower, with public savings equal to 59 percent of output. Furthermore, we find that the stark difference in optimal policy across models either persists or increases significantly in a number of sensitivity analyses.

Our results demonstrate that studying optimal policy in an infinitely lived agent model, which abstracts from the realism of a life cycle in order to render models more computationally tractable, is not without loss of generality. Not only is the optimal policy quite different when one ignores life cycle features, but the welfare consequences of ignoring them are economically significant. In the life cycle model, we find that if a government is a net borrower (as is optimal in the infinitely lived agent model) instead of being a net saver (as is optimal in the life cycle model), then an average life cycle agent would be worse off by 0.5 percent of expected lifetime consumption. Moreover, in a sensitivity exercise in which we allow agents to borrow assets, we find larger welfare losses on the order of 1 percent of expected lifetime consumption.

Two competing mechanisms generate the different optimal policies between the two models. The first, and quantitatively dominant, mechanism is the existence of a particular progression of savings through the life cycle that is absent in the infinitely lived agent model. Public debt induces a higher interest rate that encourages private saving in both models, yet has different effects on welfare in each model. In the infinitely lived agent model, public debt causes agents to live in an economy in which they have more private savings on average, thereby improving their self-insurance against idiosyncratic labor earnings risk. In contrast, in the life cycle model, although public debt increases the amount of savings agents hold during their lifetimes, agents enter the economy with little or no wealth and must accumulate savings. As a result, public debt has a smaller welfare benefit in the life cycle model because agents only realize improved self-insurance after they have accumulated savings, and furthermore must forgo early-life consumption and leisure in order to accumulate this savings. Accumulating savings at the beginning of the life mitigates enough of the welfare benefit from public debt that public saving is optimal in the life cycle model.

The differential effect of policy on income inequality in the two models is a competing, but quantitatively smaller, mechanism that reduces the divergence between the two models' optimal policies. Underlying this mechanism are three relationships: (i) increasing or decreasing public debt moves the interest rate and wage in opposite directions, (ii) income inequality is consists of inequality in both asset and labor income, and (iii) generally, asset income inequality increases with the interest rate while labor income inequality increases with the wage. Put together, these three relationships imply that optimal policy trades-off decreasing income inequality from one income source with increasing income inequality from the other. Comparing the two models, the infinitely lived agent model features a larger ratio of asset income inequality to labor income inequality compared to the life cycle model. Accordingly, in the infinitely lived agent model, a reduction in public debt improves welfare by lowering the return to saving and decreasing asset income inequality. Conversely, in the life cycle model, a decrease in public savings improves welfare by lowering the return to labor and decreasing labor income inequality. Despite its countervailing effect, we find that the inequality channel is quantitatively smaller than the effect of the accumulation phase. Thus, public savings is optimal in the life cycle model and public debt is optimal in the infinitely lived agent model.

This paper is related to an established literature that uses the standard incomplete market model with infinitely lived agents, originally developed in Bewley (1986), İmrohoroğlu (1989), Huggett (1993) Aiyagari (1994) and others, to study the optimal level of steady state government debt. In contrast to this paper, previous work has mostly utilized infinitely lived agent models and finds that public debt is optimal. Aiyagari and McGrattan (1998) is the seminal contribution to the study of optimal debt in the standard incomplete market model, and finds that public debt is optimal in an economy calibrated to resemble the U.S. Floden (2001) finds that increasing government debt can provide welfare benefits if transfers are below optimal levels. Similarly, Dyrda and Pedroni (2016) find that it is optimal for the government to be a net borrower. However, they find that optimizing both taxes and debt at the same time leads to a smaller level of optimal debt than do previous studies. Relative to these papers, we focus on how optimal policy changes when one considers a life cycle model as opposed to an infinitely lived agent model, and find that including life cycle features has large effects on optimal policy.[1]

Using variants of incomplete market models, Röhrs and Winter (2017) and Vogel (2014) also find that it can be optimal for the government to be a net saver. In both papers, the government's desire for redistribution partially explains the optimality of public savings, as public savings leads to a lower interest rate and therefore redistributes welfare from wealth-rich agents to wealth-poor agents.[2] This paper finds that the redistribution motive affects optimal policy. Yet, we find that the redistribution motive pushes optimal policy in opposing directions,

[1] Using infinitely lived agent models, Desbonnet and Weitzenblum (2012), Açikgöz (2015), Dyrda and Pedroni (2016), Röhrs and Winter (2017) find quantitatively large welfare costs of transitioning between steady states after a change in public debt. Moreover, Heinemann and Wulff (2017) demonstrate that debt-financed government stimulus after an aggregate shock can be welfare improving. We do not consider these transitional costs and instead focus on steady state comparisons to more sharply highlight the effect of the life cycle on optimal debt policy.

[2] This motive to redistribute is enhanced in both of these papers since the models are calibrated to match the upper tail of the U.S. wealth distribution, which leads to a small mass of wealth-rich agents and a larger mass of wealth-poor agents.

toward less public saving in the life cycle model and toward less public debt in the infinitely lived agent model.[3] However, we find that the existence of this accumulation phase in the life cycle model is the quantitatively dominant mechanism, leading to the optimality of public savings in the life cycle model and the optimality of public debt in the infinitely lived agent model.[4]

This paper is also related to a strand of literature that examines the effects of life cycle features on optimal fiscal policy but generally focuses on taxation instead of government debt. For example, Garriga (2001), Erosa and Gervais (2002) and Conesa et al. (2009) show that introducing a life cycle creates a motive for positive capital taxation, in contrast to the seminal findings of Judd (1985) and Chamley (1986) that optimal capital taxation is zero in the long-run of a class of infinitely lived agent models.[5] With a life cycle, if age-dependent taxation is not feasible then a positive capital tax may be optimal since it can mimic an age-dependent tax on labor income. Instead of focusing on optimal taxation in a life cycle model, this paper quantifies the effects of life cycle features on optimal government debt.[6] We find that introducing life cycle features changes optimal policy from public debt to public savings because agents must accumulate savings at the beginning of their lifetimes, not because the government would like to mimic age-dependent policy.

Finally, this paper is related to Dávila, Hong, Krusell, and Ríos-Rull (2012), whose work defines constrained efficiency in a standard incomplete markets

[3]Specifically, we find that the ratio of asset income inequality relative to lifetime labor income inequality increases with the length of the lifetime (see Dávila et al. (2012) for discussion). Thus, in the infinitely lived agent model, there is a stronger desire for the government to reduce lifetime interest income inequality which they can accomplish through public savings which lowers the interest rate. In contrast, in the life cycle model, there is more desire for the government to reduce lifetime labor income inequality, which it can accomplish by increasing public debt and thereby lowering the wage.

[4]In characterizing optimal public debt, this paper additionally abstracts from aggregate uncertainty (i.e., Barro (1979), Lucas and Stokey (1983), Aiyagari, Marcet, Sargent, and Seppala (2002), Shin (2006)), political economy distortions (i.e., Alesina and Tabellini (1990), Battaglini and Coate (2008), and Song, Storesletten, and Zilibotti (2012)) and international capital flows (i.e., Azzimonti, de Francisco, and Quadrini (2014)).

[5]In addition, Aiyagari (1995) and İmrohoroğlu (1998) demonstrate that incomplete markets can overturn the zero capital tax result with uninsurable earnings shocks and sufficiently tight borrowing constraints.

[6]Instead of isolating the effects of life cycle features on optimal debt, Garriga (2001) allows the government to choose sequences for taxes (capital, labor and consumption) as well as government debt. In contrast, our paper explicitly measures how including life cycle features alters optimal debt policy while holding other fiscal instruments constant.

model with infinitely lived agents. Constrained efficient allocations account for the effect of individual behavior on market clearing prices, while satisfying individuals' constraints. The authors show that the price system in the standard incomplete market model does not efficiently allocate resources across agents, and welfare improving equilibrium prices could be attained if agents were to systematically deviate from individually optimal savings and consumption decisions. While this paper does not characterize constrained efficient allocations, this paper's Ramsey allocation improves welfare for similar reasons: since it understands the relationship between public debt and prices, the government can implement a welfare improving allocation that individual agents cannot attain through private markets. As a result of this common mechanism, both of our papers find that a higher capital stock improves welfare. However, Dávila et al. (2012) obtains this result through matching top wealth inequality in an infinitely lived agent model, while our paper does so through adding life cycle features. In the life cycle model, the accumulation phase mitigates the welfare benefits of public debt and leads to the optimality of public savings.

The remainder of this paper is organized as follows. Section 2 illustrates the underlying mechanisms by which optimal government policy interacts with life cycle and infinitely lived agent model features. Section 3 describes the life cycle and infinitely lived agent model environments and defines equilibrium. Section 4 explains the calibration strategy, Section 5 presents quantitative results and Section 6 performs robustness exercises. Section 7 concludes.

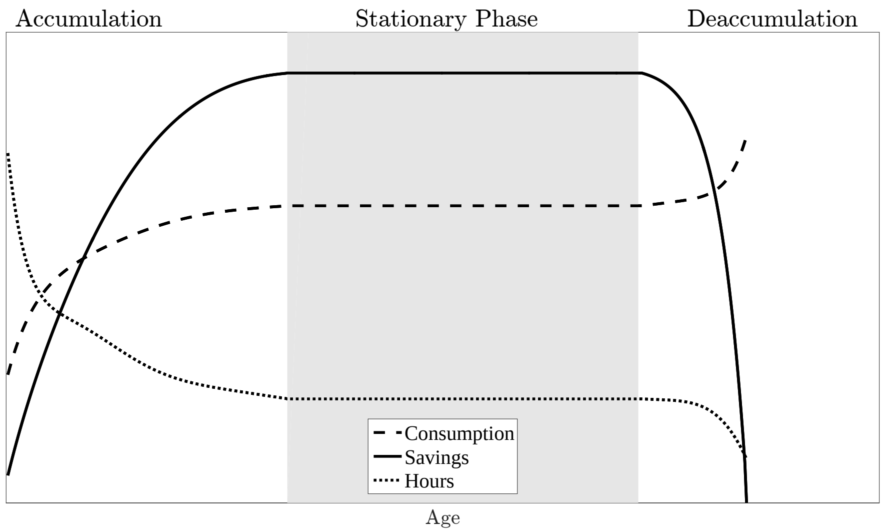
2 Illustration of the Mechanisms

In this section, we illustrate the mechanisms that lead the government to an optimal public debt or savings policy. We discuss why optimal government policy may differ in the life cycle and infinitely lived agent models.

2.1 Life Cycle Phases

In order to highlight how the life cycle may impact optimal debt policy, it will be useful to describe agents' behavior over their life cycle. To anchor the description, consider the following illustrative example. Suppose that agents are born with little or no wealth, work throughout their lifetimes and die with certainty

Figure 1: Illustrative Example of Life Cycle Phases.



within a finite number of periods. Agents face idiosyncratic labor productivity shocks and use assets to partially insure against the resulting earnings risk.

For this hypothetical economy, Figure 1 depicts cross-sectional averages for savings, hours and consumption decisions at each age. Figure 1 shows that agents experience three different phases. Agents enter the economy without any wealth and begin the *accumulation phase*, which is characterized by the accumulation of wealth for precautionary motives.[7] While accumulating a stock of savings, agents tend to work more and consume less.

Once a cohort's average wealth provides sufficient insurance against labor productivity shocks, these agents have entered the *stationary phase*.[8] This phase is characterized by savings, hours and consumption that remain constant on average.[9]

[7] Since agents do not retire from supplying labor in this simplified economy, wealth accumulation only provides self-insurance and does not finance post-retirement consumption.

[8] The stationary level of average savings is related to the "target savings level" in Carroll (1992, 1997). Given the primitives of the economy, an agent faces a tradeoff between consumption levels and consumption smoothing. The agent targets a level of savings that provides sufficient insurance while maximizing expected consumption.

[9] However, underlying constant averages for the cohort are individual agents who respond to shocks by choosing different allocations, thereby moving about various states within a non-degenerate distribution over savings, hours and consumption.

Finally, agents enter the *deaccumulation phase* as they approach the end of their lives. In order to smooth consumption in the final periods of their lives, agents attempt to deaccumulate assets so that they are not forced to consume a large quantity immediately preceding death. Furthermore, with few periods of life remaining, agents no longer want to hold as much savings for precautionary reasons. Thus, the average level of savings and labor supply decreases, while consumption increases slightly.

In comparison, infinitely lived agents only experience the equivalent of a stationary phase. On average, infinitely lived agents' consumption, hours and savings allocations remain constant.

2.2 Welfare Channels and Life Cycle Features

We identify three main channels through which public debt policy affects welfare: the *direct effect*, the *insurance channel*, and the *inequality channel*. We heuristically characterize how these channels affect optimal policy, and how these channels' effects can differ in the life cycle and infinitely lived agent models.

Direct Effect: The *direct effect* is the partial equilibrium change in the productive capital stock, aggregate consumption and aggregate output with respect to a change in public debt, when holding constant the aggregate labor supply and aggregate private savings. Mechanically, increased public debt crowds out (e.g., decreases) productive capital, thereby generating less output and decreasing aggregate consumption.[10] Generally, decreased aggregate consumption reduces welfare, which causes this mechanism to push optimal policy toward public savings. Absent any general equilibrium effects, this mechanism should operate similarly in both the life cycle and infinitely lived agent economies.

Indirect Effects: While the direct effect is a partial equilibrium effect of policy on aggregate resources, the remaining two channels affect welfare in general equilibrium, that is, by impacting market clearing prices. In particular, decreasing public savings or increasing public debt will crowd out productive capital

[10]By assuming that a representative firm operates a standard Cobb-Douglas production technology, aggregate output is a decreasing function of capital and labor inputs. Standard parameter assumptions ensure that steady state aggregate investment decreases by less than aggregate output decreases upon capital crowd out. Therefore, the resource constraint implies that aggregate consumption decreases.

and lead to an increase in the market clearing interest rate and reduction in the market clearing wage rate.

An increase in the interest rate encourages agents to save. The higher level of savings improves welfare because agents are less likely to face binding liquidity constraints and are, therefore, better insured against labor earnings risk. We refer to this channel as the *insurance channel*.

The insurance channel's welfare benefit varies substantially across the life cycle and infinitely lived agent models. In the infinitely lived agent model, agents exist in a perpetual stationary phase. This implies that public debt causes agents to live in an economy in which they have more private savings on average. Thus, increased public debt improves insurance for the average agent because he lives with more *ex ante* savings. In the life cycle model, in contrast, agents enter the economy with little or no wealth and immediately begin the accumulation phase.[11] While increased public debt may encourage agents to save more over their lifetime, agents need to accumulate this savings in the first place. Savings accumulation mitigates the welfare benefit from the insurance channel for two reasons. First, life cycle agents only realize the insurance benefit from precautionary savings after accumulating that savings. Second, the higher interest rate associated with public debt may encourage agents to accumulate a higher maximum level of savings by the time they enter the stationary phase. However, in order to reach a larger maximum level of savings, life cycle agents work more hours and consume less during the accumulation phase. Because agents prefer an intertemporally smooth allocation of consumption and hours, this more intense accumulation phase can mitigate some of the potential welfare benefit from public debt.

The second indirect channel describes the welfare effect of income inequality arising from price changes. Income inequality is composed of both asset and labor income inequality and the amount of inequality from each source increases with each source's return. Since changing public debt has opposite effects on the wage and interest rate, optimal policy trades-off decreasing income inequality from one income source with increasing income inequality from the other.

[11] If life cycle features were introduced in a dynastic model, instead of a life cycle model, where old agents bequeath wealth to agents entering the economy, then the accumulation phase may be more responsive to public policy. Consistent with Fuster, İmrohoroğlu, and İmrohoroğlu (2008), the optimal policy differences with the infinitely lived agent model could be smaller since agents would receive some initial wealth through bequests.

Therefore, the optimal tradeoff depends on the relative amount of inequality that arises from each source of income. We refer to this channel as the *inequality channel*.

Since the relative inequality deriving from labor income and asset income varies across the two models, so too will the optimal policy tradeoff. As demonstrated in Dávila, Hong, Krusell, and Ríos-Rull (2012), inequality depends on agents' lifespan. As agents live longer, lifetime labor income inequality increases because there is a greater chance that agents receive a long string of either positive or negative labor productivity shocks. However, asset income inequality will also develop because agents reduce (increase) their wealth in response to a string of negative (positive) shocks. Generally, as each agent's lifespan increases, asset income inequality increases more than labor income inequality. Accordingly, the ratio of asset income inequality to labor income inequality is larger in the infinitely lived agent model, and smaller in the life cycle model. Therefore, the inequality channel pushes optimal policy in the life cycle model toward more public debt (less public savings) and pushes optimal policy in the infinitely lived agent model toward more public savings (less public debt).

Overall, higher public debt lowers welfare through the direct channel and raises welfare through the insurance channel, while the inequality channel's effect is ambiguous. Overall, given that the direct channel and insurance channels work in opposite directions it is ambiguous whether public debt or public savings is optimal in either model.[12]. Turning to the difference in optimal policies in the two models, the inequality channel moves optimal policy towards public debt in the life cycle model while moving optimal policy towards public savings in the infinitely lived agent model. However, the benefit of public debt from the insurance channel is weaker in the life cycle model versus the infinitely lived agent model because the existence of the accumulation phase. Thus, it is unclear whether introducing the life cycle will cause optimal policy to move towards more public debt or towards public savings.. Thus, we turn to a quantitative model in order to determine the relative strength of all these effects.

[12]Moreover, since the inequality channel only dictates a difference in the relative policies in the life cycle and infinitely lived agent models, whether the inequality channel leads to public debt or public savings is unclear

3 Economic Environment

In this section, we present both the Life Cycle model and the Infinitely Lived Agent model. Given that there are many common features across models, we will first focus on the Life Cycle model in detail before providing an overview of the Infinitely Lived Agent model.

3.1 Life Cycle Model

3.1.1 Production

Assume there exist a large number of firms that sell a single consumption good in a perfectly competitive product market, purchase inputs from perfectly competitive factor markets and each operate an identical constant returns to scale production technology, $Y = ZF(K, L)$. These assumptions on primitives admit a representative firm that chooses capital (K) and labor (L) inputs in order to maximize profits, given an interest rate r, a wage rate w, a level of total factor productivity Z and capital depreciation rate $\delta \in (0, 1)$.

3.1.2 Consumers

Demographics: Let time be discrete and let each model period represent a year. Each period, the economy is inhabited by J overlapping generations of individuals. We index agents' age in the model by $j = 1, \ldots, J$, where $j = 1$ corresponds to age 21 in the data and J is an exogenously set maximum age (set to age 100 in the data). Before age J all living agents face mortality risk. Conditional on living to age j, agents have a probability s_j of living to age $j + 1$, with a terminal age probability given by $s_J = 0$. Each period a new cohort is born and the size of each successive newly born cohort grows at a constant rate $g_n > 0$.

Agents who die before age J may hold savings since mortality is uncertain. These savings are treated as accidental bequests and are equally divided across each living agent in the form of a lump-sum transfer, denoted Tr.

Preferences: Agents rank lifetime paths of consumption and labor, denoted

$\{c_j, h_j\}_{j=1}^{J}$, according to the following preferences:

$$\mathbb{E}_1 \sum_{j=1}^{J} \beta^{j-1} s_j \left[u(c_j) - v(h_j, \zeta_j') \right]$$

where β is the time discount factor. Expectations are taken with respect to the stochastic processes governing labor productivity. Furthermore, $u(c)$ and $v(h)$ are instantaneous utility functions over consumption and labor hours, respectively, satisfying standard conditions. Lastly, ζ_j' is a retirement decision that is described immediately below.

Retirement: Agents choose their retirement age, which is denoted by J_{ret}. A retired agent cannot sell labor hours and the retirement decision is irreversible. Agents choose their retirement age in the interval $j \in [\underline{J}_{ret}, \bar{J}_{ret}]$ and are forced to retire after age $\bar{J}_{ret}$. Let $\zeta_j' \equiv \mathbb{1}(j < J_{ret})$ denote an indicator variable that equals one when an agent chooses to continue working and zero upon retirement.

Labor Earnings: Agents are endowed with one unit of time per period, which they split between leisure and market labor. During each period of working life, an agent's labor earnings are we_jh_j, where w is the wage rate per efficiency unit of labor, e_j is the agent's idiosyncratic labor productivity drawn at age j and h_j is the time the agent chooses to work at age j.

Following Kaplan (2012), we assume that labor productivity shocks can be decomposed into four sources:

$$\log(e_j) = \kappa + \theta_j + \nu_j + \epsilon_j$$

where (i) $\kappa \overset{iid}{\sim} \mathcal{N}(0, \sigma_\kappa^2)$ is an individual-specific fixed effect that is drawn at birth, (ii) $\{\theta_j\}_{j=1}^{J}$ is an age-specific fixed effect, (iii) ν_j is a persistent shock that follows an autoregressive process given by $\nu_{j+1} = \rho\nu_j + \eta_{j+1}$ with $\eta \overset{iid}{\sim} \mathcal{N}(0, \sigma_\nu^2)$ and $\eta_1 = 0$, and (iv) $\epsilon_j \overset{iid}{\sim} \mathcal{N}(0, \sigma_\epsilon^2)$ is a per-period transitory shock.

For notational compactness, we denote the relevant state as a vector $\varepsilon_j = (\kappa, \theta_j, \nu_j, \epsilon_j)$ that contains each element necessary for computing contemporaneous labor earnings, $e_j \equiv e(\varepsilon_j)$, and forming expectations about future labor earnings. Denote the Markov process governing the process for ε by $\pi_j(\varepsilon_{j+1}|\varepsilon_j)$ for each ε_j, ε_{j+1} and for each $j = 1, \ldots, \bar{J}_{ret}$.

Insurance: Agents have access to a single asset, a non-contingent one-period bond denoted a_j with a market determined rate of return of r. Agents may take on a net debt position, in which case they are subject to a borrowing constraint that requires their debt position be bounded below by $\underline{a} \in \mathbb{R}$. Agents are endowed with zero initial wealth, such that $a_1 = 0$ for each agent.

3.1.3 Government Policy

The government (i) consumes an exogenous amount G, (ii) collects linear Social Security taxes τ_{ss} on all pre-tax labor income below an amount $\bar{x}$, (iii) distributes lump-sum Social Security payments b_{ss} to retired agents, (iv) distributes accidental bequests as lump-sum transfers Tr, and (v) collects income taxes from each individual.

Social Security: The model's Social Security system consists of taxes and payments. The social security payroll tax is given by τ_{ss} with a per-period cap denoted by $\bar{x}$. We assume that half of the social security contributions are paid by the employee and half by the employer. Therefore, the consumer pays a payroll tax given by: $(1/2)\ \tau_{ss} \min\{weh, \bar{x}\}$. Social security payments are computed using an averaged indexed monthly earnings (AIME) that summarizes an agents lifetime labor earnings. Following Huggett and Parra (2010) and Kitao (2014), the AIME is denoted by $\{x_j\}_{j=1}^{J}$ and is given by:

$$x_{j+1} = \left\{ \begin{array}{ll} \frac{1}{j}\left(\min\{we_j h_j, \bar{x}\} + (j-1)x_j\right) & \text{for } j \leq 35 \\ \max\left\{x_j, \frac{1}{j}\left(\min\{we_j h_j, \bar{x}\} + (j-1)x_j\right)\right\} & \text{for } j \in (35, J_{ret}) \\ x_j & \text{for } j \geq J_{ret} \end{array} \right\}$$

The AIME is a state variable for determining future benefits. Benefits consists of a base payment and an adjusted final payment. The base payment, denoted by $b^{ss}_{base}(x_{J_{ret}})$, is computed as a piecewise-linear function over the individual's average labor earnings at retirement $x_{J_{ret}}$:

$$b^{ss}_{base}(x_{J_{ret}}) = \left\{ \begin{array}{lll} \tau_{r1} & \text{for} & x_{J_{ret}} \in [0, b^{ss}_1) \\ \tau_{r2} & \text{for} & x_{J_{ret}} \in [b^{ss}_1, b^{ss}_2) \\ \tau_{r3} & \text{for} & x_{J_{ret}} \in [b^{ss}_2, b^{ss}_3) \end{array} \right\}$$

Lastly, the final payment requires an adjustment that penalizes early retirement and credits delayed retirement. The adjustment is given by:

$$b_{ss}(x_{J_{ret}}) = \left\{ \begin{array}{lll} (1 - D_1(J_{nra} - J_{ret}))b^{ss}_{base}(x_{J_{ret}}) & \text{for} & \underline{J}_{ret} \leq J_{ret} < J_{nra} \\ (1 + D_2(J_{ret} - J_{nra}))b^{ss}_{base}(x_{J_{ret}}) & \text{for} & J_{nra} \leq J_{ret} \leq \bar{J}_{ret} \end{array} \right\}$$

where $D_i(\cdot)$ are functions governing the benefits penalty or credit, $\underline{J}_{ret}$ is the earliest age agents can retire, J_{nra} is the "normal retirement age" and $\bar{J}_{ret}$ is the latest age an agent can retire.

Net Government Transfers: Taxable income is defined as labor income and capital income net of social security contributions from an employer:

$$y(h, a, \varepsilon, \zeta) \equiv \zeta w e(\varepsilon) h + r(a + Tr) - \zeta \frac{\tau_{ss}}{2} \min\{we(\varepsilon)h, \bar{x}\}$$

The government taxes each individual's taxable income according to an increasing and concave function, $\Upsilon(y(h, a, e, \zeta))$.

Define the function $T(\cdot)$ as the government's net transfers of income taxes, social security payments and social security payroll taxes to working age agents (if $\zeta = 1$) and retired agents (if $\zeta = 0$). Net transfers are given by:

$$T(h, a, \varepsilon, x, \zeta) = (1 - \zeta) b_{ss}(x) - \zeta \frac{\tau_{ss}}{2} \min\{we(\varepsilon)h, \bar{x}\} - \Upsilon(y(h, a, \varepsilon, \zeta))$$

Public Savings and Budget Balance: Each period, the government has a debt balance B and saves or borrows (denoted B') at the market interest rate r. If the government borrows then $B' < 0$ and the government repays rB' next period. If the government saves then $B' > 0$ and the government collects asset income rB'

next period. The resulting government budget constraint is:

$$G + B' - B = rB + \Upsilon_y \tag{1}$$

where Υ_y is aggregate revenues from income taxation and G is an unproductive level of government expenditures.[13] The model's Social Security system is self-financing and therefore does not appear in the governmental budget constraint.

3.1.4 Consumer's Problem

The agent's state variables consist of asset holdings a, labor productivity shocks $\varepsilon \equiv (\kappa, \theta, \nu, \epsilon)$, Social Security contribution (AIME) variable x and retirement status ζ. For age $j \in \{1, \ldots, J\}$, the agent's recursive problem is:

$$V_j(a, \varepsilon, x, \zeta) = \max_{c, a', h, \zeta'} \left[u(c) - v(h, \zeta')\right] + \beta s_j \sum_{\varepsilon'} \pi_j(\varepsilon' | \varepsilon) V_{j+1}(a', \varepsilon', x', \zeta') \tag{2}$$

$$\begin{aligned} \text{s.t.} \quad c + a' &\leq \zeta' w e(\varepsilon) h + (1+r)(a + Tr) + T(h, a, \varepsilon, x, \zeta') \\ a' &\geq \underline{a} \\ \zeta' &\in \{\mathbb{1}(j < \underline{J}_{ret}),\ \mathbb{1}(j \leq \bar{J}_{ret}) \cdot \zeta\} \end{aligned}$$

The indicator function $\mathbb{1}(j < \underline{J}_{ret})$ equals one when an agent is too young to retire and equals zero thereafter. Additionally $\mathbb{1}(j \leq \bar{J}_{ret})$ equals zero for all ages after an agent must retire and equals one beforehand. Therefore the agent's recursive problem nests all three stages of life: working life, near-retirement and retirement.[14]

[13] Two recent papers, Röhrs and Winter (2017) and Chaterjee, Gibson, and Rioja (2016) have relaxed the standard Ramsey assumption that government expenditures are unproductive. Both papers show that public savings is optimal with productive government expenditures, intuitively because there is an additional benefit to aggregate output.

[14] During an agent's working life (ages $j < \underline{J}_{ret}$) the agent's choice set for retirement is $\zeta' \in \{1, 1\}$ and therefore the agent must continue working. Near retirement (ages $\underline{J}_{ret} \leq j \leq \bar{J}_{ret}$), the agent's choice set is $\zeta' \in \{0, 1\}$ and the agent may retire by choosing $\zeta' = 0$. Lastly, if an agent has retired either because he chose retirement at a previous date ($\zeta = 0$) or because of mandatory retirement ($j > \bar{J}_{ret}$), then the choice set is $\{0, 0\}$ and $\zeta' = \zeta = 0$.

3.1.5 Recursive Competitive Equilibrium

Agents are heterogeneous with respect to their age $j \in \mathbf{J} \equiv \{1, \ldots, J\}$, wealth $a \in \mathbf{A}$, labor productivity $\varepsilon \in \mathbf{E}$, average lifetime earnings $x \in \mathbf{X}$ and retirement status $\zeta \in \mathbf{R} \equiv \{0, 1\}$. Let $\mathbf{S} \equiv \mathbf{A} \times \mathbf{E} \times \mathbf{X} \times \mathbf{R}$ be the state space and $\mathcal{B}(\mathbf{S})$ be the Borel σ-algebra on $\mathbf{S}$. Let $\mathbf{M}$ be the set of probability measures on $(\mathbf{S}, \mathcal{B}(\mathbf{S}))$. Then $(\mathbf{S}, \mathcal{B}(\mathbf{S}), \lambda_j)$ is a probability space in which $\lambda_j(S) \in \mathbf{M}$ is a probability measure defined on subsets of the state space, $S \in \mathcal{B}(\mathbf{S})$, that describes the distribution of individual states across age-j agents. Denote the fraction of the population that is age $j \in \mathbf{J}$ by μ_j. For each set $S \in \mathcal{B}(\mathbf{S})$, $\mu_j \lambda_j(S)$ is the fraction of age $j \in \mathbf{J}$ and type $S \in \mathbf{S}$ agents in the economy. We can now define a recursive competitive equilibrium of the economy.

Definition (Equilibrium): Given a government policy $(G, B, B', \Upsilon, \tau_{ss}, b_{ss})$, a *stationary recursive competitive equilibrium* is (i) an allocation for consumers described by policy functions $\{c_j, a'_j, h_j, \zeta'_j\}_{j=1}^J$ and consumer value function $\{V_j\}_{j=1}^J$, (ii) an allocation for the representative firm (K, L), (iii) prices (w, r), (iv) accidental bequests Tr, and (v) distributions over agents' state vector at each age $\{\lambda_j\}_{j=1}^J$ that satisfy:

(a) Given prices, policies and accidental bequests, $V_j(a, \varepsilon, x)$ solves the Bellman equation (2) with associated policy functions $c_j(a, \varepsilon, x, \zeta)$, $a'_j(a, \varepsilon, x, \zeta)$, $h_j(a, \varepsilon, x, \zeta)$ and $\zeta'_j(a, \varepsilon, x, \zeta)$.

(b) Given prices (w, r), the representative firm's allocation minimizes cost: $r = ZF_K(K, L) - \delta$ and $w = ZF_L(K, L)$

(c) Accidental bequests, Tr, from agents who die at the end of this period are distributed equally across next period's living agents:

$$(1 + g_n)Tr = \sum_{j=1}^{J} (1 - s_j)\mu_j \int a'_j(a, \varepsilon, x, \zeta) d\lambda_j(a, \varepsilon, x, \zeta)$$

(d) Government policies satisfy budget balance in equation (1), where aggregate income tax revenue is given by:

$$\Upsilon_y \equiv \sum_{j=1}^{J} \mu_j \int \Upsilon\left(y(h_j(a, \varepsilon, x, \zeta), a, \varepsilon, \zeta'_j(a, \varepsilon, x, \zeta))\right) \, d\lambda_j(a, \varepsilon, x, \zeta)$$

(e) Social security is self-financing:

$$\sum_{j=1}^{J}\mu_j\int\zeta'_j(a,\varepsilon,x,\zeta)\tau_{ss}\min\{we(\varepsilon)h_j(a,\varepsilon,x,\zeta),\bar{x}\}d\lambda_j(a,\varepsilon,x,\zeta)$$

$$=\sum_{j=1}^{J}\mu_j\int(1-\zeta'_j(a,\varepsilon,x,\zeta))b_{ss}(x)d\lambda_j(a,\varepsilon,x,\zeta) \tag{3}$$

(f) Given policies and allocations, prices clear asset and labor markets:

$$K-B=\sum_{j=1}^{J}\mu_j\int a\ d\lambda_j(a,\varepsilon,x,\zeta)$$

$$L=\sum_{j=1}^{J}\mu_j\int\zeta'_j(a,\varepsilon,x,\zeta)e(\varepsilon)h_j(a,\varepsilon,x,\zeta)\ d\lambda_j(a,\varepsilon,x,\zeta)$$

and the allocation satisfies the resource constraint (guaranteed by Walras' Law):

$$\sum_{j=1}^{J}\mu_j\int c_j(a,\varepsilon,x,\zeta)d\lambda_j(a,\varepsilon,x,\zeta)+G+K'=ZF(K,L)+(1-\delta)K$$

(g) Given consumer policy functions, distributions across age j agents $\{\lambda_j\}_{j=1}^{J}$ are given recursively from the law of motion $T_j^*:\mathbf{M}\to\mathbf{M}$ for all $j\in\mathbf{J}$ such that T_j^* is given by:

$$\lambda_{j+1}(\mathcal{A}\times\mathcal{E}\times\mathcal{X}\times\mathcal{R})=\sum_{\zeta\in\{0,1\}}\int_{A\times E\times X}Q_j\left((a,\varepsilon,x,\zeta),\mathcal{A}\times\mathcal{E}\times\mathcal{X}\times\mathcal{R}\right)d\lambda_j$$

where $\mathcal{S}\equiv\mathcal{A}\times\mathcal{E}\times\mathcal{X}\times\mathcal{R}\subset\mathbf{S}$, and $Q_j:\mathbf{S}\times\mathcal{B}(\mathbf{S})\to[0,1]$ is a transition function on $(\mathbf{S},\mathcal{B}(\mathbf{S}))$ that gives the probability that an age-j agent with current state $\mathbf{s}\equiv(a,\varepsilon,x,\zeta)$ transits to the set $\mathcal{S}\subset\mathbf{S}$ at age $j+1$. The transition function is given by:

$$Q_j\left((a,\varepsilon,x,\zeta),\mathcal{S}\right)=\left\{\begin{array}{ll} s_j\cdot\pi_j(\mathcal{E}|\varepsilon)^{\zeta} & \text{if } a'_j(\mathbf{s})\in\mathcal{A},\ x'_j(\mathbf{s})\in\mathcal{X},\ \zeta'_j(\mathbf{s})\in\mathcal{R} \\ 0 & \text{otherwise}\end{array}\right\}$$

where agents that continue working and transition to set $\mathcal{E}$ choose $\zeta_j'(\mathbf{s}) = 1$, while agents that transition from working life to retirement choose $\zeta_j'(\mathbf{s}) = 0$. For $j = 1$, the distribution λ_j reflects the invariant distribution $\pi_{ss}(\varepsilon)$ of initial labor productivity over $\varepsilon = (\kappa, \theta_1, 0, \epsilon_1)$.

(h) Aggregate capital, governmental debt, prices and the distribution over consumers are stationary, such that $K' = K$, $B' = B$, $w' = w$, $r' = r$, and $\lambda_j' = \lambda_j$ for all $j \in \mathbf{J}$.

3.2 Infinitely Lived Agent Model

The infinitely lived agent model differs from the life cycle model in three ways. First, agents in the infinitely lived agent model have no mortality risk ($s_j = 1$ for all $j \geq 1$) and lifetimes are infinite ($J \to \infty$). Second, labor productivity no longer has an age-dependent component ($\theta_j = \bar{\theta}$ for all $j \geq 1$). Lastly, there is no retirement ($\underline{J}_{ret} \to \infty$ such that $\zeta_j = 1$ for all $j \geq 1$) and there is no Social Security program ($\tau_{ss} = 0$ and $b_{ss}(x) = 0$ for all x).

Accordingly, we study a stationary recursive competitive equilibrium in which the initial endowment of wealth and labor productivity shocks no longer affects individual decisions and the distribution over wealth and labor productivity is time invariant.

Definition (Equilibrium): Given a government policy (G, B, B', Υ), a *stationary recursive competitive equilibrium* is (i) an allocation for consumers described by policy functions (c, a', h) and consumer value function V, (ii) an allocation for the representative firm (K, L), (iii) prices (w, r), and (v) a distribution over agents' state vector λ that satisfy:

(a) Given prices and policies, $V(a, \varepsilon)$ solves the following Bellman equation:

$$V(a,\varepsilon) = \max_{c,a',h} \left[u(c) - v(h)\right] + \beta \sum_{\varepsilon'} \pi(\varepsilon'|\varepsilon) V(a', \varepsilon') \tag{4}$$

$$\begin{aligned} \text{s.t.} \quad c + a' &\leq we(\varepsilon)h + (1+r)a + \Upsilon(y(h, a, \varepsilon)) \\ a' &\geq \underline{a} \end{aligned}$$

with associated policy functions $c(a, \varepsilon)$, $a'(a, \varepsilon)$ and $h(a, \varepsilon)$.

(b) Given prices (w, r), the representative firm's allocation minimizes cost.

(c) Government policies satisfy budget balance in equation (1), where aggregate income tax revenue is given by:

$$\Upsilon_y \equiv \int \Upsilon\big(y(h(a,\varepsilon),a,\varepsilon)\big)\; d\lambda(a,\varepsilon)$$

(d) Given policies and allocations, prices clear asset and labor markets:

$$K - B = \int a \; d\lambda(a,\varepsilon)$$

$$L = \int e(\varepsilon)h(a,\varepsilon)\; d\lambda(a,\varepsilon)$$

and the allocation satisfies the resource constraint (guaranteed by Walras' Law):

$$\int c(a,\varepsilon)d\lambda(a,\varepsilon) + G + K' = ZF(K,L) + (1-\delta)K$$

(e) Given consumer policy functions, the distribution over wealth and productivity shocks is given recursively from the law of motion $T^* : \mathbf{M} \to \mathbf{M}$ such that T^* is given by:

$$\lambda'(\mathcal{A} \times \mathcal{E}) = \int_{A \times E} Q_j\left((a,\varepsilon), \mathcal{A} \times \mathcal{E}\right) d\lambda$$

where $\mathcal{S} \equiv \mathcal{A} \times \mathcal{E} \subset \mathbf{S}$, and $Q : \mathbf{S} \times \mathcal{B}(\mathbf{S}) \to [0,1]$ is a transition function on $(\mathbf{S}, \mathcal{B}(\mathbf{S}))$ that gives the probability that an agent with current state $\mathbf{s} \equiv (a,\varepsilon)$ transits to the set $\mathcal{S} \subset \mathbf{S}$ in the next period. The transition function is given by:

$$Q\left((a,\varepsilon),\mathcal{S}\right) = \left\{ \begin{array}{cl} \pi(\mathcal{E}|\varepsilon) & \text{if } a'(\mathbf{s}) \in \mathcal{A}, \\ 0 & \text{otherwise} \end{array} \right\}$$

(f) Aggregate capital, governmental debt, prices and the distribution over consumers are stationary, such that $K' = K$, $B' = B$, $w' = w$, $r' = r$, and $\lambda' = \lambda$.

3.3 Balanced Growth Path

Following Aiyagari and McGrattan (1998), we will further assume that total factor productivity, Z, grows over time at rate $g_z > 0$. In both the life cycle model and infinitely lived agent model, we will study a balanced growth path equilibrium in which all aggregate variables grow at the same rate as output. Denote the growth rate of output as g_y. Refer to Appendix A.1 for a formal construction of the balanced growth path for this set of economies.

4 Calibration

In this section we calibrate the life cycle model and then discuss the parameter values that are different in the infinitely lived agent model. Overall, one subset of parameters are assigned values without needing to solve the model. These parameters are generally the same in both models. The other subset of parameters are estimated using a simulated method of moments procedure that minimizes the distance between model generated moments and empirical ones. We allow these parameters to vary across the models while matching the same moments in the two models. Table 1 summarizes the target and value for each parameter.

Demographics: Agents enter the economy at age 21 (or model age $j = 1$) and exogenously die at age 100 (or model age $J = 81$). We set the conditional survival probabilities $\{s_j\}_{j=1}^{J}$ according to Bell and Miller (2002) and impose $s_J = 0$. We set the population growth rate to $g_n = 0.011$ to match annual population growth in the US.

Production: Given that $Y = ZF(K, L)$, the production function is assumed to be Cobb-Douglas of the form $F(K, L) = K^{\alpha} L^{1-\alpha}$ where $\alpha = 0.36$ is the income share accruing to capital. The depreciation rate is to $\delta = 0.0833$ which allows the model to match the empirically observed investment-to-output ratio.

Preferences: The utility function is separable in the utility over consumption and disutility over labor (including retirement):

$$u(c) - v(h, \zeta') = \frac{c^{1-\sigma}}{1-\sigma} - \left(\chi_1 \frac{h^{1+\frac{1}{\gamma}}}{1+\frac{1}{\gamma}} + \zeta' \chi_2 \right) .$$

Utility over consumption is a CRRA specification with a coefficient of relative risk aversion $\sigma = 2$, which is consistent with Conesa et al. (2009) and Aiyagari and McGrattan (1998). Disutility over labor exhibits a constant intensive margin Frisch elasticity. We choose $\gamma = 0.5$ such that the Frisch elasticity consistent with the majority of the related literature as well as the estimates in Kaplan (2012).

We calibrate the labor disutility parameter χ_1 so that the cross sectional average of hours is one third of the time endowment. Finally, χ_2 is a fixed utility cost of earning labor income before retirement. The fixed cost generates an extensive margin decision through a non-convexity in the utility function. We choose χ_2 to match the empirical observation that seventy percent of the population has retired by the normal retirement age.

Labor Productivity Process: We take the labor productivity process from the estimates in Kaplan (2012) based on the estimates from the PSID data.[15] The deterministic labor productivity profile, $\{\theta_j\}_{j=1}^{\bar{J}_{ret}}$, is (i) smoothed by fitting a quadratic function in age, (ii) normalized such that the value equals unity when an agent enters the economy, and (iii) extended to cover ages 21 through 70 which we define as the last period in which agents are assumed to be able to participate in the labor activities ($\bar{J}_{ret}$).[16] The permanent, persistent, and transitory idiosyncratic shocks to individual's productivity are normally distributed with zero mean. The remaining parameters are also set in accordance with the Kaplan's (2012) estimates: $\rho = 0.958$, $\sigma_\kappa^2 = 0.065$, $\sigma_\nu^2 = 0.017$ and $\sigma_\epsilon^2 = 0.081$.

Government: Consistent with Aiyagari and McGrattan (1998) we set government debt equal to two-thirds of output. We set government consumption equal to 15.5 percent of output consistent. This ratio corresponds to the average of government expenditures to GDP from 1998 through 2007.[17]

[15]For details on estimation of this process, see Appendix E in Kaplan (2012). A well known problem with a log-normal income process is that the model generated wealth inequality does not match that in the data, primarily due to missing the fat upper tail of the distribution. However, Röhrs and Winter (2017) demonstrate that when the income process in an infinitely lived agent model is altered to match the both the labor earnings and wealth distributions (quintiles and gini coefficients), the change in optimal policy is relatively small, with approximately 30 percentage points due to changing the income process and the remaining 110 percentage points due to changing borrowing limits, taxes and invariant parameters (such as risk aversion, the Frisch elasticity, output growth rate and depreciation).

[16]The estimates in Kaplan (2012) are available for ages 25-65.

[17]We exclude government expenditures on Social Security since they are explicitly included in our model.

Income Taxation: The income tax function and parameter values are from Gouveia and Strauss (1994). The functional form is:

$$\Upsilon(y) = \tau_0 \left(y - \left(y^{-\tau_1} + \tau_2 \right)^{-\frac{1}{\tau_1}} \right)$$

The authors find that $\tau_0 = 0.258$ and $\tau_1 = 0.768$ closely match the U.S. tax data. When calibrating the model we set τ_2 such that the government budget constraint is satisfied.

Social Security: We set the normal retirement age to 66. Consistent with the minimum and maximum retirement ages in the U.S. Social Security system, we set the interval in which agents can retire to the ages 62 and 70. The early retirement penalty and later retirement credits are set in accordance with the Social Security program. In particular, if agents retire up to three years before the normal retirement age agents benefits are reduced by 6.7 percent for each year they retire early. If they choose to retire four or five years before the normal retirement age benefits are reduced by an additional 5 percent for these years. If agents choose to delay retirement past normal retirement age then their benefits are increased by 8 percent for each year they delay. The marginal replacement rates in the progressive Social Security payment schedule $(\tau_{r1}, \tau_{r2}, \tau_{r3})$ are also set at their actual respective values of 0.9, 0.32 and 0.15. The bend points where the marginal replacement rates change (b_1^{ss}, b_2^{ss}, b_3^{ss}) and the maximum earnings ($\bar{x}$) are set equal to the actual multiples of mean earnings used in the U.S. Social Security system so that b_1^{ss}, b_2^{ss} and $b_3^{ss} = \bar{x}$ occur at 0.21, 1.29 and 2.42 times average earnings in the economy. We set the payroll tax rate, τ_{ss} such that the program's budget is balanced. In our baseline model the payroll tax rate is 10.3 percent, roughly equivalent with the statutory rate.[18]

Infinitely Lived Agent Model: The infinitely lived agent model does not have an age-dependent wage profile. For comparability across models, we replace the age-dependent wage profile with the population-weighted average of θ_j's, such that $\bar{\theta} = \sum_{j=1}^{J_{ret}} (\mu_j / \sum_{j=1}^{J_{ret}} \mu_j)\theta_j \approx 1.86$.[19] In the absence of a retirement decision,

[18] Although the payroll tax rate in the U.S. economy is slightly higher than our calibrated value, the OASDI program includes additional features outside of the retirement benefits.

[19] When calibrating the stochastic process for idiosyncratic productivity shocks, we use the same process in the both the life cycle and infinitely lived agent models. Using the same underly-

Table 1: Calibration Targets and Parameters for Baseline Economy.

Description	Parameter	Value	Target or Source
Demographics			
Maximum Age	J	81 (100)	By Assumption
Min/Max Retirement Age	$\underline{J}_{ret}, \bar{J}_{ret}$	43, 51 (62, 70)	Social Security Program
Population Growth	g_n	1.1%	Conesa et al (2009)
Survival Rate	$\{s_j\}_{j=1}^{J}$	—	Bell and Miller (2002)
Preferences and Borrowing			
Coefficient of RRA	σ	2.0	Kaplan (2012)
Frisch Elasticity	γ	0.5	Kaplan (2012)
Coefficient of Labor Disutility	χ_1	55.3	Avg. Hours Worked = 1/3
Fixed Utility Cost of Labor	χ_2	1.038	70% retire by NRA
Discount Factor	β	1.012	Capital/Output = 2.7
Borrowing Limit	$\underline{a}$	0	By Assumption
Technology			
Capital Share	α	0.36	NIPA
Capital Depreciation Rate	δ	0.0833	Investment/Output = 0.255
Productivity Level	Z	1	Normalization
Output Growth	g_y	1.85%	NIPA
Labor Productivity			
Persistent Shock, autocorrelation	ρ	0.958	Kaplan (2012)
Persistent Shock, variance	σ_{ν}^2	0.017	Kaplan (2012)
Permanent Shock, variance	σ_{κ}^2	0.065	Kaplan (2012)
Transitory Shock, variance	σ_{ϵ}^2	0.081	Kaplan (2012)
Mean Earnings, Age Profile	$\{\theta\}_{j=1}^{\bar{J}_{ret}}$	—	Kaplan (2012)
Government Budget			
Government Consumption	G/Y	0.155	NIPA Average 1998-2007
Government Savings	B/Y	-0.667	NIPA Average 1998-2007
Marginal Income Tax	τ_0	0.258	Gouveia and Strauss (1994)
Income Tax Progressivity	τ_1	0.786	Gouveia and Strauss (1994)
Income Tax Progressivity	τ_2	4.541	Balanced Budget
Social Security			
Payroll Tax	τ_{ss}	0.103	Social Security Program
SS Replacement Rates	$\{\tau_{ri}\}_{i=1}^{3}$	See Text	Social Security Program
SS Replacement Bend Points	$\{b_i^{ss}\}_{j=1}^{3}$	See Text	Social Security Program
SS Early Retirement Penalty	$\{\kappa_i\}_{i=1}^{3}$	See Text	Social Security Program

we set $\chi_2 = 0$. Lastly, we recalibrate the parameters (β, χ_1) to the same targets

ing process will imply that cross-sectional wealth inequality will be different across the two models. One reason is that the life cycle model will have additional cross-sectional inequality due to the humped shaped savings profiles, which induces the accumulation, stationary, and deaccumulation phases. We view these difference in inequality as a fundamental difference between the two models and, therefore, choose not to specially alter the infinitely lived agent model to match a higher level of cross-sectional inequality.

as in the life cycle model and choose τ_2 to balance the government's budget.

5 Quantitative Effects of the Life Cycle on Optimal Policy

This section computes optimal public debt policy in the life cycle and infinitely lived agent models and quantifies the contribution of the life cycle to policy differences. We quantify the effect of each life cycle feature through the construction of a series of counterfactual models that systematically removes life cycle features until recovering the infinitely lived agent model. Therefore the counterfactual models isolate the response of optimal policy to each life cycle model component. We conclude this section by computing the welfare gains from implementing optimal public debt policy and further highlighting the effect of life cycle features on optimal by decomposing those welfare gains.

5.1 Optimal Public Policy

In both the life cycle and infinitely lived agent models, the government is a benevolent Ramsey planner that fully commits to fiscal policy. The planner maximizes social welfare by choosing a budget feasible level of public savings ($B > 0$) or public debt ($B < 0$) subject to allocations being a stationary recursive competitive equilibrium. We consider an ex-ante Utilitarian social welfare criterion that evaluates the expected utility of an agent in the steady state economy.[20]

For the life cycle model, the Ramsey planner chooses public savings to maximize the expected lifetime utility of newborn agents as follows,

$$S_J(V_1, \lambda_1) \equiv \max_B \left\{ \int V_1(a, \varepsilon, x, \zeta; B)\, d\lambda_1(a, \varepsilon, x, \zeta; B) \quad \text{s.t.} \quad (1),\ (3) \right\}$$

where the value function $V_1(\cdot; B)$, distribution function $\lambda_1(\cdot; B)$ and policy functions embedded in equations (1) and (3) are determined in competitive equilibrium and depend on the planner's choice of public savings. Furthermore, $B' = B$ in steady state. Since the distribution of taxable income and tax revenues

[20] Our analysis focuses on welfare across steady states. This analysis omits the transitional costs between steady states which can be large. See Domeij and Heathcote (2004), Fehr and Kindermann (2015) and Dyrda and Pedroni (2016) for a discussion of these transitional costs.

depend on public savings, we adjust the income tax parameter τ_0 and the payroll tax rate τ_{ss} to ensure that the government budget is balanced and Social Security is self-financing.[21]

For the infinitely lived agent model, the Ramsey planner chooses public savings to maximize the expected utility of infinitely lived agents as follows,

$$S_\infty(V,\lambda) \equiv \max_B \left\{ \int V(a,\varepsilon;B)\, d\lambda(a,\varepsilon;B) \quad \text{s.t.} \quad G = rB + \Upsilon_y(\tau_0,B) \right\}$$

The welfare maximization problem is nearly identical to that of the life cycle model's, except that the value function and distribution function do not depend on age and there is no Social Security program, so that equation (3) does not define the feasible set.

We find that the two models generate starkly different optimal policies, which are reported in Table 2. In the infinitely lived agent model, the government is a net borrower with optimal public debt equal to 24 percent of output.[22] On the other hand, in the life cycle model, the government's optimal policy is public savings equal to 59 percent of output. Thus, including life cycle features causes optimal policy to switch from public debt to savings, with approximately an 85 percentage point swing in optimal policy.

We quantify the welfare gain from implementing the optimal policy in each economy. In particular, we compute *consumption equivalent variation (CEV)* – the percent of lifetime consumption that a life cycle model agent would be willing to pay ex ante – from inhabiting an economy with an optimal public savings policy of 59 percent of output instead of an economy with the infinitely lived agent model's optimal public debt policy of 24 percent of output. We find that the 85 percentage point difference in optimal policies corresponds to a welfare gain of 0.45 percent of expected lifetime consumption. The welfare gain is economically significant, demonstrating that ignoring life cycle features when determining

[21] We choose to use τ_0 to balance the government budget instead of the other income taxation parameters (τ_1, τ_2) so that the average income tax rate is used to clear the budget, as opposed to changing in the progressivity of the income tax policy. The average tax rate is the closest analogue to the flat tax that Aiyagari and McGrattan (1998) use to balance the government's budget in their model.

[22] This is generally consistent with Aiyagari and McGrattan's (1998) optimal policy. The differences in optimal policy are due to this paper assumes a different stochastic process governing labor productivity, a different utility function, non-linear income taxation and different parameter values.

Table 2: Aggregates and Prices Across Models

	Life Cycle		Infinitely Lived	
	Baseline	**Optimal**	**Baseline**	**Optimal**
Public Savings/Output	**-0.67**	**0.59**	**-0.67**	**-0.24**
Consumption	0.53	0.54	0.66	0.66
Output	0.93	1.01	1.16	1.18
Labor	0.53	0.54	0.66	0.67
Productive Capital	2.50	3.03	3.13	3.23
Private Savings	3.12	2.43	3.90	3.51
Public Savings	-0.62	0.6	-0.77	-0.28
Interest Rate	5.0%	3.6%	5.0%	4.8%
Wage	1.12	1.19	1.12	1.13

optimal debt policy can have nontrivial welfare effects.

5.2 The Effect of Life Cycle Features on Optimal Policy

The 85 percentage point difference in optimal policies is due to the three main differences between the life cycle and infinitely lived agent models: (i) agents in the life cycle model experience all three life cycle phase, including an accumulation phase, while agents in the infinitely lived agent model experience a perpetual stationary phase, (ii) other age-dependent features in the life cycle model, such as mortality risk, an age-dependent wage profile, retirement and Social Security, do not exist in the infinitely lived agent model and (iii) agents' lifespan differ between the two models.

In order to characterize the individual effects of these three differences on optimal policy between the life cycle and infinitely lived agent models, starting with the life cycle model we compute optimal policy in two counterfactual economies that systematically remove life cycle features.[23] The first counterfactual model is the "No Age-Dependent Features" economy, which is similar to the life cycle model but excludes all age-dependent features (e.g., mortality risk, age-dependent wage profile, retirement and Social Security system), while

[23] In order to make quantitative comparisons across models, each counterfactual model's parameters are recalibrated to match all relevant the targets described in Section 4.

Table 3: Optimal Public Savings-to-Output (Percent)

	Counterfactual Models		
Life Cycle	**No Age Features**	**Infinitely Lived with Accumulation**	**Infinitely Lived**
59%	200%	248%	-24%

maintaining the maximum lifespan of $J = 80$ periods. The second counterfactual economy, the "Infinitely Lived with Accumulation" economy, also eliminates age-dependent model features but additionally extends agents' lifespan to $J = 1000$ periods. We choose $J = 1000$ because it is sufficiently large such that for a newborn the expected present value of the flow of utility from the end of the lifetime is essentially zero.. Therefore, the "Infinitely Lived with Accumulation" economy approximates the ex ante expected lifetime utility of an infinitely lived agent, yet agents have finite lifespans. However, since agents still enter the economy with no wealth, this economy is an approximation to the infinitely lived agent economy that still includes an accumulation phase.

Table 3 reports optimal policies for the life cycle, counterfactual, and infinitely lived agent models. First, comparing the life cycle model and "No Age-Dependent Features" economy isolates the effect of age-dependent features, which primarily leads to an increase in the expected working lifetime due to removing retirement and mortality. We find that the optimal public savings changes from 59 to 200 percent of output. Comparing the "No Age-Dependent Features" and "Infinitely Lived with Accumulation" counterfactual economies isolates the effect of further increasing agents' lifespan from 80 periods to an approximation of an infinite lifespan. This effect additionally increases optimal savings from 200 to 248 percent of output. Finally, comparing the "Infinitely Lived with Accumulation" economy with the infinitely lived agent model isolates the effect of the accumulation phase on optimal policy, which changes optimal policy from public savings to public debt equal to 24 percent of output.

Comparing optimal policies across these four models yields two notable results. First, more public savings is optimal in the "No Age-Dependent" and "Infinitely Lived with Accumulation" counterfactual economies' than in the life cycle model. Removing life cycle features creates counterfactual models that become increasingly similar to the infinitely lived agent model, yet optimal policy

Table 4: Effect of Lifespan on Inequality (Coefficient of Variation)

	Life Cycle	**No Age Features**	**Infinitely Lived w/ Accumulation**	**Infinitely Lived**
Asset Income Inequality	0.64	0.62	0.85	0.98
Labor Income Inequality	0.30	0.27	0.27	0.31
$\frac{\textbf{Asset Income Inequality}}{\textbf{Labor Income Inequality}}$	2.11	2.30	3.11	3.15

diverges from that in the infinitely lived agent model. In particular, removing life cycle features generates more optimal public savings relative to the life cycle model, instead of public debt (or less public savings) as is optimal in the infinitely lived agent model. Second, by comparing optimal policies from the "Infinitely Lived with Accumulation" economy and the infinitely lived agent model, we observe that removing the accumulation phase accounts for a 275 basis point change in optimal policy, changing from public savings to public debt. These results highlight two competing mechanisms that determine the difference in optimal policy between the life cycle and infinitely lived agent models: (i) the differential effect of the inequality channel across models, and (ii) the effect of the accumulation phase, which is absent from infinitely lived agent model.

The inequality channel has a differential effect on optimal policy in the two models because the amount of labor income inequality relative to asset income inequality generally depends on agents' lifespan. As agents live and work longer, asset income inequality tends to increase because there is more time for labor productivity shocks to propagate into the wealth distribution and enlarge the difference in wealth between lucky and unlucky agents. Relative to the life cycle model, agents in the "No Age-Dependent" and "Infinitely Lived with Accumulation" counterfactual models work for a longer length of time (e.g., due to removing mortality and retirement, or mechanically extending lifespan). Table 4 confirms that with the longer working lifetime, asset income inequality relative to both labor income inequality is larger in the counterfactual economies than it is in the standard life cycle model under the baseline public debt policy of 67% of output.

Since government policy affects the returns from labor and capital in opposite directions, optimal policy trades off reducing income inequality from the source

Table 5: Lifetime Total Income Inequality (Coefficient of Variation)

	Life Cycle	No Age Features	Infinitely Lived w/ Accumulation	Infinitely Lived
Baseline Policy	0.38	0.33	0.31	0.47
Optimal Policy	0.36	0.30	0.28	0.45
Percent Change	-4.4%	-8.5%	-10.0%	-3.2%

for which the factor price decreases with increasing income inequality from the source for which the factor price increases. Thus, the counterfactual models have higher levels of optimal public savings than does the life cycle model, because asset income inequality rises relative to labor income inequality as we remove life cycle features and extend agents' expected working lifetimes. The change in total lifetime income inequality measures in Table 5 confirm that, in fact, adopting optimal policy reduces total income inequality.[24] Thus, the inequality channel causes the optimal level of public savings to increase after eliminating life cycle features but retaining the accumulation phase.

The second mechanism, the existence of the accumulation phase in the life cycle model, is the reason why public savings is optimal in the life cycle model instead of public debt. Comparing optimal policies in the "Infinitely Lived with Accumulation" with the infinitely lived agent models isolates the effect of accumulation phase, which leads to 275 percentage point difference in optimal policy (as reported in Table 3). To further isolate this effect, we conduct a computational experiment in which we compute the optimal policy of the "Infinitely Lived with Accumulation" counterfactual model according to an alternative social welfare function that only incorporates the expected present value of utility after a given age $j^* > 1$, and ignores the flow of utility from ages 1 to $j^* - 1$. Thus, as j^* increases, the social welfare function ignores more of the accumulation phase.[25]

The computational experiments demonstrates that as the accumulation phase

[24] In the baseline life cycle model, in which asset income inequality is relatively smaller compared to labor income inequality, we find that adopting public savings reduces total income inequality. However, we find in Section 5.3 that in terms of welfare, this change increases inequality.

[25] Specifically, government policy maximizes agents' expected lifetime utility as of age j^*, subject

matters less for social welfare, optimal policy tends toward more public debt, as shown in Figure 2. The left panel in Figure 2 plots the optimal policy of the "Infinitely Lived with Accumulation" counterfactual model under the alternative welfare criterion, while the right panel plots the percentage of the accumulation phase that is ignored when computing optimal policy. On the x-axis, both graphs vary the percent of lifetime that the threshold age j^* represents. We observe that optimal policy monotonically decreases from public savings of approximately 250 percent of GDP when all of the lifetime is considered, to an optimal public debt policy when the social welfare function ignores at least 5.2 percent of agents' early lifetime, or approximately 70 percent of the accumulation phase.[26]

To summarize, we find that the existence of an accumulation phase mitigates the welfare benefit from public debt. However, extending agents' working lifetime increases the amount of asset income inequality relative to labor income inequality, thereby increasing the welfare benefit from public debt. Thus, when comparing the life cycle and the infinitely lived agent models, the existence of age-dependent features and a shorter lifespan drive optimal policy toward public debt, while the existence of the accumulation phase drives optimal policy toward public savings. Overall, we find that the effects of the accumulation phase dominate the effects of other life cycle model features on optimal policy, ultimately resulting in the optimality of public savings in the life cycle model.

5.3 Welfare Decomposition

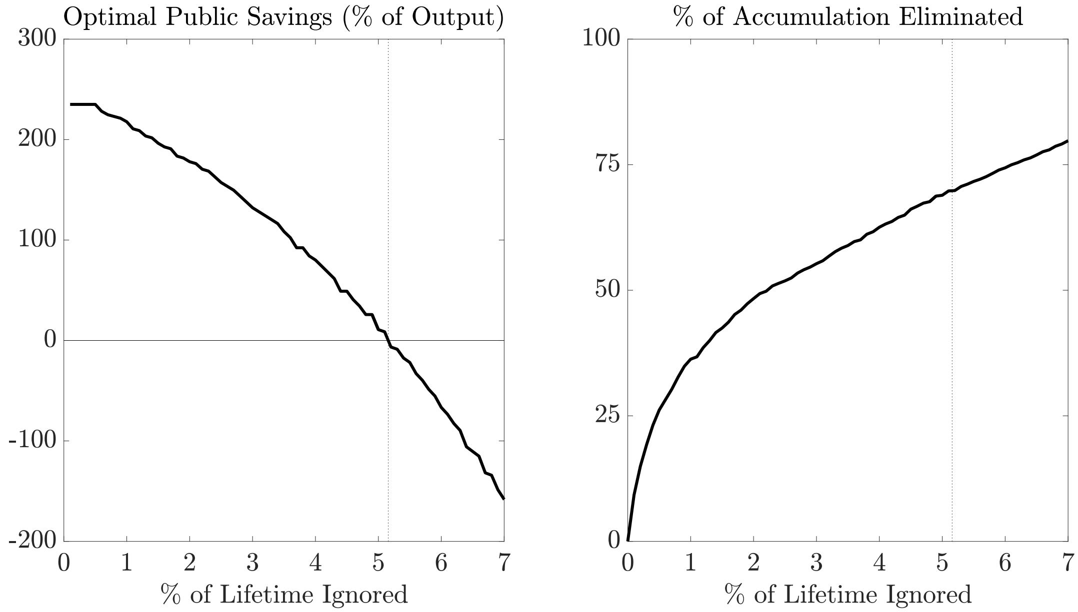
In order to further understand why public debt is optimal in the infinitely lived agent model and public savings is optimal in the life cycle model we examine the welfare implications from adopting public savings instead of public debt.

to allocations being determined in competitive equilibrium, as follows:

$$\tilde{S}(V_{j^*}, \lambda_{j^*}) \equiv \max_B \Big\{ \int V_{j^*}(a, \varepsilon; B)\, d\lambda_{j^*}(a, \varepsilon; B) \quad \text{s.t.} \quad G = rB + \Upsilon_y(\tau_0, B) \Big\}.$$

[26]Similarly, we find public savings is optimal in the infinitely lived agent model when the government only considers the welfare agents that closely resemble life cycle agent entering the accumulation phase. Specifically, if the government only places Pareto weight on the set of borrowing constrained agents with the median persistent component of the labor productivity shock, then optimal public savings equals 300 percent of output. This result reinforces the notion that wealth accumulation reduces the welfare benefit from public debt.

Figure 2: Optimal Policy and Eliminating Accumulation



Notes: The left panel graphs the optimal public savings to output ratio (y-axis) associated with ignoring a given percent of early life utility flows (x-axis). The percent of "Lifetime Ignored" is measured as $100 \cdot (j^*/J)$, using the given value of j^* and $J = 1000$. The right panel graphs the percent of accumulation that is eliminated under the optimal policy associated with ignoring a given percent of early life utility flows. The percent of eliminated wealth accumulation is defined as the average private savings of j^*-age agents relative to the peak average savings and converted to a percent, given a particular optimal public savings policy. The vertical dashed line demarcates the percent of early lifetime utility ignored at which optimal policy switches from public savings to debt.

Specifically, in both models, we quantify the welfare effects from a 85 percentage point change in policy, from the optimal public debt policy in the infinitely lived agent model to the optimal public savings policy in the life cycle model.

The welfare effects from changing public policy reflect the change in aggregate resources available to agents and the allocation of those resources across agents and across their lifetimes. Thus, we decompose the consumption equivalent variation (denoted Δ_{CEV}) into a *level effect* (Δ_{level}), an *age effect* (Δ_{age}) and a *distribution effect* (Δ_{distr}). For each of these components, we further decompose into welfare effects induced by changes in consumption and hours (for example, the consumption level effect and hours level effect are denoted $\Delta_{C_{level}}$ and $\Delta_{H_{level}}$,

Table 6: Welfare Decompositions

	Life Cycle (% Change)	**Infinitely Lived (% Change)**
Overall CEV	0.45	-0.05
Level (Δ_{level})	-0.12	-0.16
Consumption ($\Delta_{C_{level}}$)	1.42	0.92
Hours ($\Delta_{H_{level}}$)	-1.52	-1.06
Age (Δ_{age})	0.67	0
Consumption ($\Delta_{C_{age}}$)	0.27	0
Hours ($\Delta_{H_{age}}$)	0.40	0
Distribution (Δ_{distr})	-0.09	0.11
Consumption ($\Delta_{C_{distr}}$)	-0.18	-0.20
Hours ($\Delta_{H_{distr}}$)	0.09	0.31

Notes: The life cycle and infinitely lived agent model welfare decompositions compare allocations under a 24% public debt-to-output and a 59% public savings-to-output ratio.

respectively).[27] The decomposition is defined as

$$(1+\Delta_{CEV}) = \underbrace{\left[(1+\Delta_{C_{level}})(1+\Delta_{H_{level}})\right]}_{\equiv(1+\Delta_{level})} \cdot \underbrace{\left[(1+\Delta_{C_{age}})(1+\Delta_{H_{age}})\right]}_{\equiv(1+\Delta_{age})} \cdot \underbrace{\left[(1+\Delta_{C_{distr}})(1+\Delta_{H_{distr}})\right]}_{\equiv(1+\Delta_{distr})}.$$

The level effect captures the welfare change for a fictitious "representative agent," absent any idiosyncratic or lifecycle variation in consumption or hours. The age effect measures agents' change in welfare as a result of changing age-specific average levels of consumption and hours, net of changes in aggregate consumption and hours. Accordingly, the age effect captures the welfare effect of a change in the slope of the average consumption and hours age-profiles. Note that the age effect does not exist in the infinitely lived agent model and therefore

[27]More generally, we follow Floden (2001) in characterizing four components of the CEV: a level effect (Δ_L), an insurance effect (Δ_I), a redistribution effect (Δ_R) and a labor hours effect (Δ_H). We combine the insurance and redistribution effects to form the "distribution effect". Lastly, we add two elements to Floden's (2001) original decomposition. First, we add an age effect, which only exists in the life cycle model. Second, because an extensive margin retirement decision is unique to the life cycle model, we incorporate both intensive and extensive margins into the welfare decomposition of the life cycle model's hours allocation. Appendix A.2 formally derives the decomposition.

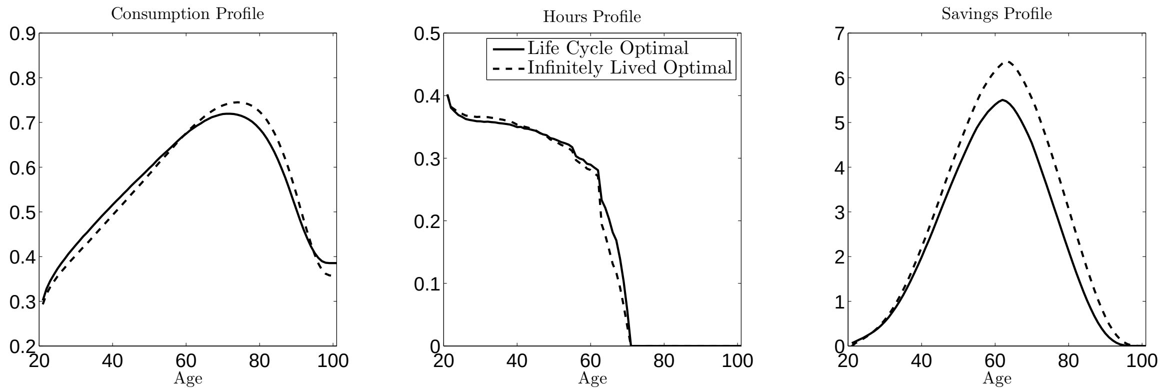
infinitely lived agents attain zero welfare change through age effects. Lastly, the distribution effects measures the remaining change in welfare that results from a change in the distribution of consumption and hours across agents.

The welfare decomposition demonstrates that the age effect is crucial in explaining why public savings increases welfare in the life cycle model and decreases welfare in the infinitely lived agent model. The 0.45 percent welfare improvement from implementing public savings in the life cycle model is due to a 0.67 percent increase from the age effect that is partially offset by a 0.12 percent decreases and 0.09 percent decrease from the level and distribution effects, respectively. In contrast, the small 0.05 percent CEV loss in the infinitely lived agent model can be attributed to approximately offsetting level and distribution effects, featuring a 0.16 percent decrease from the level effect and a 0.11 increase from the distribution effect.

The life cycle model's positive age effect from adopting public savings indicates an improved allocation of consumption and hours across ages. Agents possess standard concave utility functions and prefer smooth consumption and hours allocations. As shown in Figure 3, the lower interest rate associated with adopting public savings policy leads agents to amass savings less quickly during the accumulation phase such that they enjoy a more equal allocation of consumption and leisure over their lifecycle. In contrast, the accumulation phase does not exist in the infinitely lived agent model and public savings will discourage precautionary savings. Thus, without the benefit of the age effect in the infinitely lived agent model public savings reduces welfare. These results confirm that the existence of the accumulation phase in the life cycle model, and its absence from the infinitely lived agent model, explain the stark difference in optimal policy between the two models.

Finally, the distribution effect from adopting a public savings policy partially offsets the level and age effects in each model, leading to welfare reduction in the life cycle model and a welfare improvement in the infinitely lived agent model. The difference distribution effect across models corresponds to the inequality channel. A higher wage and lower interest rate from public savings has different effects on inequality in the life cycle and infinitely lived agent models. As discussed in Section 2.2, a longer working lifetime in the infinitely lived agent model leads to more asset income inequality relative to labor earnings inequality. Thus, a higher wage and lower interest rate can reduce existing total income

Figure 3: Life Cycle Model: Consumption, Savings and Hours Profiles



Notes: Solid lines are cross-sectional averages for consumption, savings, and hours by age in the life cycle economy under its optimal public savings policy. The dashed lines are cross-sectional averages for the suboptimal debt policy from the infinitely lived agent economy.

inequality. In the life cycle model, the opposite holds true; since asset income inequality relative to labor income inequality is smaller, a lower interest rate and higher wage exacerbates lifetime total income inequality.[28]

To summarize, the welfare effects highlight the two competing mechanisms that lead to different optimal policy across the life cycle and infinitely lived agent models. First, the accumulation phase mitigates the welfare benefit from public debt in the life cycle model, which is measured by the age effect and is only present in the life cycle model. Second, the inequality channel decreases the difference in optimal policies across models, since adopting public savings increases welfare inequality in the life cycle model but reduces it in the infinitely lived agent model. On net, we find the quantitative magnitude of the age effect dominates the distribution effect.

[28]In contrast to the age and distribution effect, the level effect from adopting public savings is similar in both models. In particular, there is a welfare increase from the consumption level effect and a welfare decrease from the hours level effect. Public savings leads to more productive capital so both output and consumption increase. However, the larger stock of productive capital leads to a higher wage which encourages more labor. Overall, the disutility from more labor dominates the increase in utility from more consumption because the lower interest rate associated with public savings reduces the incentives for agents to save so they are more likely to face binding liquidity constraints (i.e. a reduction in the benefit from the insurance channel).

6 Robustness

In this section we check how sensitive our main results are along three dimensions of the models: matching aggregate labor supply instead of aggregate hours, allowing agents to borrow, and including a more realistic distribution of wealth. We find that our results are robust to each of these model changes.

6.1 Aggregate Labor Supply

In Section 4, we calibrated the coefficient on the disutility of labor in both models, χ_1, to ensure that agents choose to work one third of their *available time endowment*, on average. In the life cycle model, we define this endowment as agents' available time prior to the normal retirement age. However, because agents do not retire in the infinitely lived agent model, the available time endowment is larger and leads to an approximately 25% larger aggregate labor supply under the baseline calibration (see Table 2). Our baseline calibration, while standard, implicitly treats the difference in effective labor supply across models as a defining characteristic of the life cycle model.

In order to test whether this difference in aggregate labor is important for our results, we recompute optimal policy in the infinitely lived agent model under an alternative calibration that makes aggregate labor supply consistent across the two models. Specifically, we now calibrate χ_1 in the infinitely lived agent model to match 27.8 percent, which is the percent of the *total lifetime time endowment* that agents work in the life cycle model.[29] The recalibration reduces the infinitely lived agent model's baseline aggregate labor from $L = 0.66$ to $L = 0.55$, which is quite close to that in the life cycle model.[30]

Under this recalibration, optimal public debt in the infinitely lived agent model equals 31 percent of output. The alternative calibration leads to a fairly small, 10% increase in optimal debt compared to the benchmark calibration. Therefore, changing the calibration target for labor supply does not alter the result that adding a life cycle has large effects on optimal policy.

[29] Using this target, we find that $\chi_1 = 72.6$ and $\beta = 0.967$.

[30] By construction, aggregate hours are the same in both models under the baseline calibration. However, small differences remain in aggregate labor. This is because although the average hours decisions are the same, the distribution of hours across the productivity distributions are different in the two models. This is partly because of the age-specific human capital in the life cycle model, which is absent in the infinitely lived agent model.

6.2 Borrowing

In Section 4, we assumed that agents could not borrow. However, the insurance channel operates by endogenously relaxing borrowing constraints and, furthermore, more strongly impacts optimal policy in the infinitely lived agent model. In order to test how sensitive the difference in optimal policy (and, subsequently, the insurance channel) across models is to the no borrowing assumption, we exogenously relax borrowing constraints in both models and recompute optimal policy. Specifically, we set a new borrowing limit at 30 percent of each economy's aggregate private savings under the baseline calibration, while holding all other model parameters fixed.

We find that it is still optimal for the government in the infinitely lived agent model to be a net borrower with public debt equal to 24 percent of output. In contrast, we find that it is optimal for the government in the life cycle model to be a net saver with public savings equal to 118 percent of output, which is twice as much public savings as the optimal policy with a no borrowing constraint. Therefore, allowing for borrowing increases the discrepancy between the optimal policies in the two models.

When borrowing is allowed in the life cycle model, young agents who enter the economy with zero wealth and experience a sharply increasing average labor productivity profile, $\{\theta_j\}_{j=1}^{\bar{J}_{ret}}$, over the majority of their working lifetime tend to initially borrow against future income, instead of accumulate savings. Borrowing helps agents intertemporally smooth consumption and insure against idiosyncratic shocks. Agents then wait until later in life, when their labor productivity is expected to be relatively high, to repay their debt and become net savers. Accordingly, higher public savings decreases the interest rate, which simultaneously encourages relatively greater early-life consumption and makes debt repayment less costly.

While life cycle agents' incentives to borrow derive from their increasing average labor productivity profile, infinitely lived agents experience a constant average labor productivity profile, $\theta_j = \bar{\theta}$ for all j. While persistent shocks (ν) in both models generate expected labor productivity growth conditional on having low labor productivity, a constant average labor productivity profile reduces the mass of agents experiencing labor productivity growth. This reduces the average agent's incentive to borrow relative to the life cycle model. As a result, we find a minimal effect on optimal policy from allowing borrowing for these infinitely lived agents. Therefore, the main mechanism by which government debt improves welfare in the infinitely lived agent model is robust to changes in borrowing limits.

7 Conclusion

This paper characterizes the effect of a life cycle on optimal public debt and evaluates the mechanisms by which a life cycle affects optimal policy. We find that the optimal policies are strikingly different between life cycle and infinitely lived agent models. We find that it is optimal for the government to be a *net saver* with savings equal to 59% of output when life cycle features are included. In contrast, it is optimal for the government to be a *net debtor* with debt equal to 24% of output when these life cycle features are excluded.

Furthermore, there are economically significant welfare consequences from not accounting for life cycle features when determining the optimal policy. We find that if a government implemented the infinitely lived agent model's optimal 24% debt-to-output policy in the life cycle model, then life cycle agents would be worse off by nearly one-half percent of expected lifetime consumption.

We have shown that the existence of the accumulation phase is the predominant reason for the drastically different optimal policies between the life cycle and infinitely lived agent models. In the infinitely lived agent model, higher public debt implies that an average agent begins each period of time with more savings and is, therefore, better insured against labor earnings risk. In the life cycle model, in contrast, agents enter the economy with little or no wealth and must accumulating savings. While a higher level of public debt might encourage life cycle agents to hold more savings during their lifetime, the fact that agents must accumulate this savings stock mitigates the welfare benefits from public debt.

When using quantitative models to answer economic questions, economists constantly face a trade-off between tractability and realism. Our results demonstrate that when examining the welfare consequences of public debt, it is not without loss of generality to utilize the more tractable infinitely lived agent model instead of a life cycle model.

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A Appendix

A.1 Construction of the Balanced Growth Path

We construct the Balanced Growth Path in multiple parts. First we construct the Balanced Growth Path using aggregates from the models. Then, we construct the Balanced Growth Path using individual agents' allocations. The last two sections develop the Balanced Growth Path for any features unique to the infinitely lived agent or life cycle models.

A.1.1 Aggregate Conditions

Balanced Growth Path: A Balanced Growth Path (BGP) is a sequence

$$\{C_t, A_t, Y_t, K_t, L_t, B_t, G_t\}_{t=0}^{\infty}$$

such that (i) for all $t = 0, 1, \ldots$ $C_t, A_t, Y_t, K_t, B_t, G_t$ grow at a constant rate g,

$$\frac{Y_{t+1}}{Y_t} = \frac{C_{t+1}}{C_t} = \frac{A_{t+1}}{A_t} = \frac{K_{t+1}}{K_t} = \frac{B_{t+1}}{B_t} = \frac{G_{t+1}}{G_t} = 1 + g$$

(ii) per capita variables all grow at the same constant rate g_w:

$$\frac{Y_{t+1}/N_{t+1}}{Y_t/N_t} = \frac{C_{t+1}/N_{t+1}}{C_t/N_t} = \frac{A_{t+1}/N_{t+1}}{A_t/N_t} = \frac{K_{t+1}/N_{t+1}}{K_t/N_t} = \frac{B_{t+1}/N_{t+1}}{B_t/N_t} = \frac{G_{t+1}/N_{t+1}}{G_t/N_t} = 1 + g_w$$

and (iii) hours worked per capita are constant:

$$\frac{L_{t+1}}{N_{t+1}} = \frac{L_t}{N_t} = \frac{L_0}{N_0}$$

Denote time 0 variables without a time subscript, for example $L \equiv L_0$.

Growth Rates: Let all growth derive from TFP $g_z > 0$ and population $g_n > 0$ growth. Then on a balanced growth path we assume:

$$Z_t = (1 + g_z)^t Z$$

$$N_t = (1 + g_n)^t N$$

where Z and N are steady state values. Then, from part (iii) of the definition, growth in labor is:

$$\frac{L_{t+1}}{L_t} = \frac{L_{t+1}/N_{t+1}}{L_t/((1 + g_n)N_t)} = 1 + g_n$$

In steady state $Y = ZK^{\alpha}L^{1-\alpha}$. Let output growth be given by $g > 0$. Therefore the production function gives:

$$Y_t = Z_t K_t^{\alpha} L_t^{1-\alpha} \quad \Longrightarrow \quad (1 + g) = (1 + g_z)^{\frac{1}{1-\alpha}}(1 + g_n)$$

Lastly, from parts (*ii*) and (*iii*) of the Balanced Growth Path definition, we can solve for the growth of per capita variables:

$$\frac{Y_{t+1}/N_{t+1}}{Y_t/N_t} = \frac{Z_{t+1}}{Z_t}\left(\frac{K_{t+1}/N_{t+1}}{K_t/N_t}\right)^{\alpha}\left(\frac{L_{t+1}/N_{t+1}}{L_t/N_t}\right)^{1-\alpha} \implies (1+g_w) = (1+g_z)^{\frac{1}{1-\alpha}}$$

Prices: From Euler's theorem we know:

$$Y_t = \alpha Y_t + (1-\alpha)Y_t = (r_t+\delta)K_t + w_t L_t$$

Accordingly, the wage and interest rate depend on the capital-labor ratio. Growth in the capital-labor ratio is:

$$\frac{K_{t+1}/L_{t+1}}{K_t/L_t} = (1+g_z)^{\frac{1}{1-\alpha}} = 1+g_w$$

Therefore, the growth rate for the wage is:

$$\frac{w_{t+1}}{w_t} = \frac{Z_{t+1}}{Z_t}\cdot\left(\frac{K_{t+1}/L_{t+1}}{K_t/L_t}\right)^{\alpha} = 1+g_w$$

and the growth rate for the interest rate is:

$$\frac{r_{t+1}+\delta}{r_t+\delta} = \frac{Z_{t+1}}{Z_t}\cdot\left(\frac{K_{t+1}/L_{t+1}}{K_t/L_t}\right)^{\alpha-1} = 1$$

Therefore wages grow while interest rates do not.

Equilibrium Conditions: The detrended *asset market clearing condition* is:

$$K_t = A_t + B_t \quad\implies\quad K = A - B$$

The detrended *resource constraint* is:

$$C_t + K_{t+1} + G_t = Y_t + (1-\delta)K_t \quad\implies\quad C + (g+\delta)K + G = Y$$

and the detrended *government budget constraint* is:

$$G_t + rB_t = T_t + B_{t+1} - B_t \quad\implies\quad G + (r-g)B = T$$

A.1.2 Individual Conditions

Preferences: We assume that labor disutility has a time-dependent component. Specifically, we assume labor disutility grows at the same rate as the utility over consumption, such that $v_{t+1}(h) = (1+g_w)^{1-\sigma} v_t(h)$. Therefore, total utility is:

$$U_t(c_t, h_t) = u(c_t) - v_t(h_t) = \left[(1+g_w)^{1-\sigma}\right]^t (u(c) - v(h)).$$

Social Security: In order for the AIME to grow at the same rate as the wage, we assume a cost of living adjustment (COLA) on Social Security taxes and payments. For social security taxes, the cap on eligible income grows at the rate of wage growth, $\bar{x}_t = (1+g_w)^t \bar{x}$. Furthermore, base payment bend points $b^{ss}_{i,t} = (1+g_w)^t b^{ss}_i$ and base payment values $\tau_{r,i,t} = (1+g_w)^t \tau_{r,i}$ for $i = 1, 2, 3$.

Tax Function: On a Balanced Growth Path, (c_t, a'_{t+1}, a_t) and $\tilde{y}_t$ must all grow at the same rate as the wage. Furthermore, the tax function must grow at the same rate as the wage. Recalling the tax function, $\Upsilon_t(\tilde{y}_t)$, τ_2 must grow at the same rate as $\tilde{y}_t^{-\tau_1}$. Rewrite as:

$$\Upsilon_t(\tilde{y}_t) = \tau_0 \left((1+g_w)^t \tilde{y} - \left([(1+g_w)^t]^{-\tau_1} \tilde{y}^{-\tau_1} + [(1+g_w)^t]^{-\tau_1} \tau_2 \right)^{-\frac{1}{\tau_1}} \right) = (1+g_w)^t \, \Upsilon(\tilde{y})$$

Individual Budget Constraint: An agent's time t budget constraint is:

$$c_t + a'_{t+1} \leq w_t \varepsilon_t h_t + (1+r_t) a_t - T_t(\cdot)$$
$$c + (1+g_w) a' \leq w \varepsilon h + (1+r) a - T(\cdot)$$

where $\{c, a', a, h, w, r, \varepsilon\}$ are stationary variables. Given that the tax function $\Upsilon(\tilde{y})$ grows at rate g_w, so will the transfer function $T(h, a, \varepsilon)$ in the infinitely lived agent model. Furthermore, given that the Social Security program $\{\bar{x}, b^{ss}_i, \tau_{r,i}\}$ grows at rate g_w, so will the transfer $T(h, a, \varepsilon, x, \zeta')$ function in the life cycle model.

A.1.3 Life Cycle Model

Individual Problem: On the balanced growth path of the life cycle model, the stationary dynamic program is:

$$V_j(a,\varepsilon,x,\zeta) = \max_{c,a',h,\zeta'} \ \left[u(c) - \zeta' v(h)\right] + [\beta s_j (1+g_w)^{1-\sigma}] \sum_{\varepsilon'} \pi_j(\varepsilon'|\varepsilon) V_{j+1}(a',\varepsilon',x',\zeta')$$

$$\begin{aligned} \text{s.t.} \quad c + (1+g_w)a' &\leq \zeta' w e(\varepsilon) h + (1+r)(a+Tr) + T(h,a,\varepsilon,x,\zeta') \\ a' &\geq \underline{a} \\ \zeta' &\in \{\mathbb{1}(j < \underline{J}_{ret}),\ \mathbb{1}(j \leq \bar{J}_{ret}) \cdot \zeta\} \end{aligned}$$

Distributions: For j-th cohort at time t, the measure over (a,ε,x,ζ) is given by:

$$\begin{aligned} \lambda_{j,t}(a_t,\varepsilon,x_t,\zeta) &= \lambda_{j,t-1}\left(\frac{a_t}{1+g_w},\varepsilon,\frac{x_t}{1+g_w},\zeta\right)(1+g_n) \\ &= \lambda_{j,t-m}\left(\frac{a_t}{(1+g_w)^m},\varepsilon,\frac{x_t}{(1+g_w)^m},\zeta\right)(1+g_n)^m \quad \forall\ m \leq t \\ &= \lambda_j(a,\varepsilon,x,\zeta) N_{t-j+1}. \end{aligned}$$

Therefore, $\lambda_j(a,\varepsilon,x,\zeta)$ is a stationary distribution over age j agents that integrates to one.

Aggregation: Aggregate consumption in the life cycle model is constructed as follows. Define the relative size of cohorts as $\mu_1 = 1$ and:

$$\mu_{j+1} = \frac{N_{t-j}}{N_t} \cdot \prod_{i=1}^{j} s_i = (1+g_n)^{-j} \prod_{i=1}^{j} s_i = \frac{s_j \mu_j}{1+g_n} \quad \forall\ j = 1,\ldots,J-1$$

Let $C_{j,t}$ be aggregate consumption per age-j agent, which is derived from the age-j agent's allocation:

$$C_{j,t} = \int (1+g_w)^t c_j(a,\varepsilon,x,\zeta) d\lambda_j = (1+g_w)^t \int c_j(a,\varepsilon,x,\zeta) d\lambda_j = (1+g_w)^t C_j$$

where C_j is the stationary aggregate consumption per age-j agent. Accordingly, aggregate consumption is:

$$
\begin{aligned}
C_t &= N_t\Big(C_{1,t} + s_1(1+g_n)^{-1}C_{2,t} + \cdots + \left(\prod_{i=1}^{J-1} s_i\right)(1+g_n)^{-(J-1)}C_{J,t}\Big) \\
&= (1+g_w)^t N_t \sum_{j=1}^{J} \mu_j C_j \\
&= (1+g)^t C
\end{aligned}
$$

where C is the stationary level of aggregate consumption and where we have normalized $N = 1$.

We can similarly construct the remaining aggregates $\{A, K, Y, B, G\}$ on the balanced growth path. Notably, however, labor per capita does not grow. Aggregate labor per capita is constructed as:

$$
L_t = N_t \sum_{j=1}^{J} \mu_j L_j \quad \Longrightarrow \quad L = \frac{L_t}{N_t} = \sum_{j=1}^{J} \mu_j \int \zeta_j'(a,\varepsilon,x,\zeta)\varepsilon h_j(a,\varepsilon,x,\zeta) d\lambda_j
$$

which is the sum over ages of aggregate labor per age-j agent.

A.1.4 Infinitely Lived Agent Model

In order to isolate the effects on optimal policy due to fundamental differences in the life cycle and infinitely lived agent models, and not due to differences in balanced growth path constructs, we want sources of output growth (e.g. TFP and population growth) to be consistent across models. Thus, we incorporate population growth into the infinitely lived agent model. To be consistent with the life cycle model, we construct a balanced growth path in which the infinitely lived agent model's income and wealth distributions grow homothetically. Our representation of this growth concept is consistent with a dynastic model in which population growth arises from agents producing offspring and valuing the utility of their offspring.

To elaborate in more detail, two additional assumptions admit a balanced growth path with population growth. First, agents exogenously reproduce at rate g_n and next period's offspring are identical to each other. Second, the parent values each offspring identically, and furthermore values each offspring as much as they value their self. Formally, if the parent has continuation value $\beta\mathbb{E}[v(a', \varepsilon')]$, then the parent values all its offspring with total value of $g_n\beta\mathbb{E}[v(a', \varepsilon')]$.

These two assumptions imply two features. First, each offspring is identical to its parent. That is, if the parent's state vector is (a', ε') next period, then so is each offspring's state vector. As a result, the value function of each offspring upon birth is $v(a', \varepsilon')$. Second, since the parent values each offspring equal to its own continuation value, it is optimal for the parent to save save $(1 + g_n)a'$ in total. The portion $g_n a'$ is bequeathed to offspring, and the portion a' is kept for next period.

Individual Problem: On the balanced growth path of the Infinitely Lived Agent Model, the stationary dynamic program is then:

$$v(a, \varepsilon) = \max_{c,a',h} U(c, h) + [\beta(1 + g_w)^{1-\sigma}](1 + g_n) \sum_{\varepsilon'} \pi(\varepsilon'|\varepsilon)v(a', \varepsilon')$$

$$\text{s.t.} \qquad c + (1 + g_n)(1 + g_w)a' \leq w\varepsilon h + (1 + r)a - T(y)$$

where $y \equiv w\varepsilon h + ra$ and optimality conditions are given by:

$$\chi v'(h) = u'(c)w\varepsilon(1 - T'(y))$$

$$u'(c) = \beta(1 + g_w)^{-\sigma}(1 + r) \sum_{\varepsilon'} \pi(\varepsilon'|\varepsilon)u'(c')(1 - T'(y')).$$

Notice that the optimality conditions do not change relative to a world without population growth. However, the cost of savings has increased since agents bequeath wealth to offspring.

Distribution: The distribution evolves according to:

$$\lambda_{t+1}(a_{t+1}, \varepsilon_{t+1}) = \sum_{\varepsilon_t} \pi(\varepsilon_{t+1}|\varepsilon_t) \int_A \mathbb{1}\left[a'_{t+1}(a_t, \varepsilon_t) = a_{t+1}\right] \lambda_t(a_t, \varepsilon_t)da_t$$

The stationary distribution $\lambda(a, \varepsilon)$ has measure 1 over $\mathcal{A} \times \mathcal{E}$ but the mass of agents grows at rate g_n:

$$\lambda_t(a_t, \varepsilon) = \lambda_{t-1}\left(\frac{a_t}{1 + g_w}, \varepsilon\right)(1 + g_n)$$

$$= \lambda_{t-s}\left(\frac{a_t}{(1+g_w)^s}, \varepsilon\right)(1+g_n)^s \quad \forall\, s \le t$$

$$= \lambda(a,\varepsilon) N_t$$

Therefore, applying the transformation above and normalizing by N_{t+1} yields:

$$\lambda(a', \varepsilon') = \sum_{\varepsilon} \pi(\varepsilon'|\varepsilon) \int_A \mathbb{1}\left[a'(a,\varepsilon) = a'\right] \frac{\lambda(a,\varepsilon)}{1+g_n} da$$

Aggregation: To construct aggregate consumption, wealth, savings and labor, multiply individual allocations by the size of the population (N_t) and sum using the stationary distribution λ. For example, aggregate consumption is:

$$C_t = N_t \int (1+g_w)^t c(a,\varepsilon) d\lambda = (1+g)^t \int c(a,\varepsilon) d\lambda = (1+g)^t C$$

We can similarly construct the remaining aggregates $\{A, K, Y, B, G\}$ on the balanced growth path. Notably, however, aggregate labor per capita does not grow:

$$\frac{L_t}{N_t} = \int \varepsilon h(a,\varepsilon) d\lambda$$

where again $N_0 = 1$ by normalization.

A.2 Welfare Decomposition

Proposition 1: *If preferences are additively separable in utility over consumption, $u(c)$, and disutility over hours and working life, $v(h,\zeta)$, then welfare changes can be decomposed as:*

$$(1+\Delta_{CEV}) = \underbrace{\left[(1+\Delta_{C_{level}})(1+\Delta_{H_{level}})\right]}_{\equiv(1+\Delta_{level})} \cdot \underbrace{\left[(1+\Delta_{C_{age}})(1+\Delta_{H_{age}})\right]}_{\equiv(1+\Delta_{age})} \cdot \underbrace{\left[(1+\Delta_{C_{distr}})(1+\Delta_{H_{distr}})\right]}_{\equiv(1+\Delta_{distr})}.$$

Proof: Consider two economies, $i \in \{1,2\}$. Define ex ante welfare in economy $i \in \{1,2\}$ derived from consumption, hours and retirement allocations $\{c^i_j(\mathbf{s}), h^i_j(\mathbf{s}), \zeta^i_j(\mathbf{s})\}_{j=1}^J$ over states $\mathbf{s} \equiv (a, \varepsilon, x, \zeta)$ distributed with $\lambda^i_j(\mathbf{s})$ as:

$$S^i = U(c^i) - V^h(h^i) - V^\zeta(\zeta^i)$$

where

$$U(c^i) \equiv \int \mathbb{E}_0 \left[\sum_{j=1}^{J} \beta^{j-1} s_j u\left(c_j^i\right) \right] d\lambda_1^i$$

$$V^h(h^i) \equiv \int \mathbb{E}_0 \left[\sum_{j=1}^{J} \beta^{j-1} s_j v\left(h_j^i\right) \right] d\lambda_1^i$$

$$V^\zeta(\zeta^i) \equiv \int \mathbb{E}_0 \left[\sum_{j=1}^{J} \beta^{j-1} s_j \chi_2 \zeta_j^i \right] d\lambda_1^i.$$

Denote the Consumption Equivalent Variation (CEV) by Δ_{CEV}, which is defined as the percent of expected lifetime consumption that an agent inhabiting economy $i = 1$ would pay *ex ante* in order to inhabit economy $i = 2$:

$$(1 + \Delta_{CEV})^{1-\sigma} U(c^1) - V^h(h^1) - V^\zeta(\zeta^1) = U(c^2) - V^h(h^2) - V^\zeta(\zeta^2).$$

First we decompose the CEV into levels, age and distribution effects for consumption allocations. The overall consumption effect is:

$$(1 + \Delta_C)^{1-\sigma} U(c^1) - V^h(h^1) - V^\zeta(\zeta^1) = U(c^2) - V^h(h^1) - V^\zeta(\zeta^1).$$

For the level effect, note that aggregate consumption in economy i is

$$C^i = \sum_{j=1}^{J} \mu_j \int_{\mathbf{S}} c_j^i(\mathbf{s}) d\lambda_j^i(\mathbf{s}).$$

We follow Conesa et al. (2009) in defining the level effect, in a different but equivalent way to Floden (2001), by

$$(1 + \Delta_{C_{level}})^{1-\sigma} U(c^1) - V^h(h^1) - V^\zeta(\zeta^1) = U(\left(C^2/C^1\right) c^1) - V^h(h^1) - V^\zeta(\zeta^1).$$

For the age effect, note that age-specific average consumption in economy i is

$$C_j^i = \int_{\mathbf{S}} c_j^i(\mathbf{s}) d\lambda_j^i(\mathbf{s}),$$

and utility over age-cohort average level of consumption at each age is,

$$U(C_j^i) \equiv \int \mathbb{E}_0 \left[\sum_{j=1}^{J} \beta^{j-1} s_j \int u\left(C_j^i\right) \right] d\lambda_1^i.$$

Then define the consumption age effect in economy i as

$$U\left((1-\omega_{C_{age}}^i)C^i\right) = U(C_j^i),$$

such that

$$(1-\omega_{C_{age}}^i) = \left(\frac{\sum_{j=1}^{J} \beta^{j-1} s_j u\left(C_j^i\right)}{\left[\sum_{j=1}^{J} \beta^{j-1} s_j\right] u(C^i)} \right)^{\frac{1}{1-\sigma}},$$

which gives the overall consumption age effect,

$$(1+\Delta_{C_{age}}) \equiv \frac{1-\omega_{C_{age}}^2}{1-\omega_{C_{age}}^1} = \frac{\left(\sum_{j=1}^{J} \beta^{j-1} s_j u\left(C_j^2\right)\right)^{\frac{1}{1-\sigma}} / C^2}{\left(\sum_{j=1}^{J} \beta^{j-1} s_j u\left(C_j^1\right)\right)^{\frac{1}{1-\sigma}} / C^1}.$$

Lastly, again following Floden (2001) and Conesa et al. (2009), define the consumption distribution effect in economy i as the residual of the overall consumption effect:

$$U\left((1-\omega_{distr}^i)C_j^i\right) = U\left(c^2\right)$$

such that

$$(1-\omega_{C_{distr}}^i) = \left(\frac{\sum_{j=1}^{J} \beta^{j-1} s_j u(c^i)}{\sum_{j=1}^{J} \beta^{j-1} s_j u(C_j^i)} \right)^{\frac{1}{1-\sigma}},$$

which gives the overall consumption distribution effect,

$$(1+\Delta_{C_{distr}}) \equiv \frac{1-\omega_{C_{distr}}^2}{1-\omega_{C_{distr}}^1} = \frac{\left(U(c^2)/U(C_j^2)\right)^{\frac{1}{1-\sigma}}}{\left(U(c^1)/U(C_j^1)\right)^{\frac{1}{1-\sigma}}}.$$

The consumption effect decomposition is verified as follows,

$$\begin{aligned}(1+\Delta_C) &= (1+\Delta_{C_{level}}) \quad \cdot \quad (1+\Delta_{C_{age}}) \quad \cdot \quad (1+\Delta_{C_{distr}}) \\ &= (C^2/C^1) \quad \cdot \quad \frac{(U(C_j^2)/U(C_j^1))^{\frac{1}{1-\sigma}}}{C^2/C^1} \quad \cdot \quad \frac{(U(c^2)/U(c^1))^{\frac{1}{1-\sigma}}}{(U(C_j^2)/U(C_j^1))^{\frac{1}{1-\sigma}}}\end{aligned}$$

Likewise we define the overall hours effect as

$$(1+\Delta_H)^{1-\sigma}U(c^2) - V^h(h^1) - V^\zeta(\zeta^1) = U(c^2) - V^h(h^2) - V^\zeta(\zeta^2)$$

which implies that

$$(1+\Delta_H) = \left(\frac{U(c^2) - V^h(h^2) - V^\zeta(\zeta^2) + V^h(h^1) + V^\zeta(\zeta^1)}{U(c^2)}\right)^{\frac{1}{1-\sigma}} = \frac{1+\Delta_{CEV}}{1+\Delta_C}$$

For the level effect, note that aggregate hours and the mass of working agents in economy i is

$$H^i = \sum_{j=1}^{J} \mu_j \int_{\mathbf{S}} h_j^i(\mathbf{s}) d\lambda_j^i(\mathbf{s}),$$

$$I^i = \sum_{j=1}^{J} \mu_j \int_{\mathbf{S}} \zeta_j^i(\mathbf{s}) d\lambda_j^i(\mathbf{s}).$$

We follow Conesa et al. (2009) in defining the hours level effect. However, since our economy features both an intensive and extensive margin labor decision, we simultaneously decompose welfare arising from hours and retirement decisions,

$$(1+\Delta_{H_{level}})^{1-\sigma}U(c^2) - V^h(h^1) - V^\zeta(\zeta^1) = U(c^2) - V^h\left(\frac{H_2}{H_1}h^1\right) - V^\zeta\left(\frac{I^2}{I^1}\zeta^1\right)$$

For the age effect, note that age-specific average hours and average mass of working agents in economy i is

$$H_j^i = \int_{\mathbf{S}} h_j^i(\mathbf{s}) d\lambda_j^i(\mathbf{s}),$$

$$I_j^i = \int_{\mathbf{S}} \zeta_j^i(\mathbf{s}) d\lambda_j^i(\mathbf{s}),$$

and expected lifetime utility over age-cohort average level of hours and working at each age is,

$$V^h(H_j^i) \equiv \int \mathbb{E}_0 \left[\sum_{j=1}^{J} \beta^{j-1} s_j v \left(H_j^i \right) \right] d\lambda_1^i,$$

$$V^\zeta(I_j^i) \equiv \int \mathbb{E}_0 \left[\sum_{j=1}^{J} \beta^{j-1} s_j \chi_2 I_j^i \right] d\lambda_1^i,$$

Then define the hours age effect in economy i as

$$(1+\Delta_{H_{age}})^{1-\sigma} U(c^2) - V^h \left(\frac{H_2}{H_1} h^1 \right) - V^\zeta \left(\frac{I^2}{I^1} \zeta^1 \right) = U(c^2) - V^h \left(\frac{V^h(H_j^2)}{V^h(H_j^1)} h^1 \right) - V^\zeta \left(\frac{V^\zeta(I_j^2)}{V^\zeta(I_j^1)} \zeta^1 \right)$$

Following Floden (2001) and Conesa et al. (2009), the hours distribution effect is then a residual of the overall hours effect:

$$(1+\Delta_{H_{distr}})^{1-\sigma} U(c^2) - V^h \left(\frac{V^h(H_j^2)}{V^h(H_j^1)} h^1 \right) - V^\zeta \left(\frac{V^\zeta(I_j^2)}{V^\zeta(I_j^1)} \zeta^1 \right) = U(c^2) - V^h(h^2) - V^\zeta(\zeta^2)$$

The decomposition can be verified using a first order approximation of the $i = 2$ allocation around the $i = 1$ allocation and therefore a first order approximation of $\Delta_{H_{level}}, \Delta_{H_{age}}, \Delta_{H_{distr}}$ around zero. Note that $u'(c)c/u(c) = (1-\sigma)$, $v'(h)h/v(h) = 1 + 1/\gamma$ and that since $\log(1+\Delta) \approx \Delta$, then

$$\log(1+\Delta_H) = \log(1+\Delta_{H_{level}}) + \log(1+\Delta_{H_{age}}) + \log(1+\Delta_{H_{distr}})$$

implies

$$\Delta_H \approx \Delta_{H_{level}} + \Delta_{H_{age}} + \Delta_{H_{distr}}.$$

The first order approximations yield,

$$\Delta_H \approx \frac{1}{1-\sigma} \left(\frac{U(c^2) - V^h(h^2) - V^\zeta(\zeta^2) + V^h(h^1) + V^\zeta(\zeta^1)}{U(c^2)} - 1 \right)$$

$$\Delta_{H_{level}} \approx \left(\frac{1+\frac{1}{\gamma}}{1-\sigma}\right)\left(1-\frac{H^2}{H^1}\right)\frac{V^h(h^1)}{U(c^2)} + \frac{1}{1-\sigma}\left(1-\frac{I^2}{I^1}\right)\frac{V^\zeta(\zeta^1)}{U(c^2)}$$

$$\Delta_{H_{age}} \approx \left(\frac{1+\frac{1}{\gamma}}{1-\sigma}\right)\left(\frac{H^2}{H^1}-\frac{V^h(H_j^2)}{V^h(H_j^1)}\right)\frac{V^h(h^1)}{U(c^2)} + \frac{1}{1-\sigma}\left(\frac{I^2}{I^1}-\frac{V^\zeta(I_j^2)}{V^\zeta(I_j^1)}\right)\frac{V^\zeta(\zeta^1)}{U(c^2)}$$

and

$$\Delta_{H_{distr}} \approx -\left(\frac{1+\frac{1}{\gamma}}{1-\sigma}\right)\left(1-\frac{V^h(H_j^2)}{V^h(H_j^1)}\right)\frac{V^h(h^1)}{U(c^2)} - \frac{1}{1-\sigma}\left(1-\frac{V^\zeta(I_j^2)}{V^\zeta(I_j^1)}\right)\frac{V^\zeta(\zeta^1)}{U(c^2)}$$
$$+\frac{1}{1-\sigma}\left(\frac{U(c^2)-V^h(h^2)+V^h(h^1)-V^\zeta(h^2)+V^\zeta(h^1)}{U(c^2)}-1\right)$$

Verification comes from regrouping terms in the distribution effect:

$$\Delta_{H_{distr}} \approx -(\Delta_{H_{level}}+\Delta_{H_{age}})+\Delta_H$$

$$\Delta_H \approx \Delta_{H_{level}}+\Delta_{H_{age}}+\Delta_{H_{distr}}$$

as desired. ■