Lecture 6

(1) General Equilibrium: Existence and Computation

(2) General Equilibrium: Extensions

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Econ 606: Adv. Topics in Macroeconomics
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Agenda

- **Last Time (2/20/18): Buffer Stock Savings Model**
  - Marginal Propensity to Consume
  - Income process with permanent and transitory shocks
  - Quantitative Implications
- **Today: General Equilibrium**
  - Neoclassical Growth Model
  - Formulation with uninsurable idiosyncratic labor risk
  - To Do: prove existence, discuss computation
  - Is government debt a good or bad thing?

General Equilibrium

Basic Idea

- Study version of growth model
- Representative firm
  - Demands capital (savings) and labor
- Include heterogeneous agents:
  - Idiosyncratic labor productivity shocks
  - Access to incomplete bond markets
  - Supply labor and capital (savings)
- Interest rate r endogenously determined

$$\boxed{\text{Savings(r)} = \text{Investment(r)}}$$

Consumers

Familiar:

- Unit continuum of ex-ante identical agents
- Preferences: $u' > 0$, $u'' < 0$, in DARA class (e.g. CRRA)
- Access to non-contingent bond with interest r

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New Assumptions:

- More detail w.r.t. labor supply and labor markets

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- For computational tractability:
  - Assume discreteness: $\varepsilon \in E \equiv \{\underline{\varepsilon}, \ldots, \overline{\varepsilon}\}$
  - Markov transition probabilities: $\pi(\varepsilon'|\varepsilon)$

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- Assume a Law of Large Numbers:
  - $\pi(\varepsilon'|\varepsilon)$ is the population measure of $\varepsilon'|\varepsilon$
  - $\pi^*(\varepsilon) \equiv$ fraction of population with ε shock
- (Effective) Labor Supply:

$$L = \sum_{\varepsilon \in E} \pi^*(\varepsilon)\varepsilon$$

- Labor supply is exogenous

Consumers

Consumer's Problem:

$$v(a,\varepsilon) = \max_{c,a'} u(c) + \beta \sum_{\varepsilon' \in E} \pi(\varepsilon'|\varepsilon) v(a',\varepsilon')$$

$$\begin{aligned} \text{s.t.} \quad c + a' &\leq w\varepsilon + (1+r)a \\ a' &\geq \underline{a} \end{aligned}$$

Euler Equation:

$$u_c(c) \geq \beta(1+r) \sum_{\varepsilon' \in E} \pi(\varepsilon'|\varepsilon) u_c(c') \quad \text{w.e. if } a' > \underline{a}$$

Policy Functions:

- Denote $a'(a,e)$ and $c(a,\varepsilon)$

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  - Large number of firms
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- Cost Minimization:

$$r = \alpha z \left(K/L\right)^{\alpha-1} - \delta \qquad \& \qquad w = (1-\alpha) z \left(K/L\right)^{\alpha}$$

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- L is in fixed supply, can rewrite:

$$K(r) \equiv \left(\frac{\alpha z}{r+\delta}\right)^{\frac{1}{1-\alpha}} L \qquad \& \qquad w(r) \equiv (1-\alpha) z \left(\frac{K(r)}{L}\right)^{\alpha}$$

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  - Capital Supplied?
  - Need to aggregate consumers' savings
  - What is the measure of consumers for each level of savings?

Distributions

Transition Function: $P : S \times \mathbf{B} \to [0,1]$

- Let $S \equiv A \times E$ s.t. $a \in A \equiv [\underline{a}, \overline{a}]$ and $\varepsilon \in E$
- Probability an agent with state (a, ε) transitions into set $\mathcal{A} \times \mathcal{E}$

$$P\left((a,\varepsilon), \mathcal{A} \times \mathcal{E}\right) = \mathbf{1}\left[a'(a,\varepsilon) \in \mathcal{A}\right] \sum_{\varepsilon' \in \mathcal{E}} \pi(\varepsilon'|\varepsilon)$$

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Distribution

- $\lambda(a, \varepsilon) \equiv$ joint distribution over wealth and income
- Fixed point mapping: $T^* : \Lambda(S, \mathbf{B}) \to \Lambda(S, \mathbf{B})$

$$(T^*\lambda)(\mathcal{A} \times \mathcal{E}) = \int_{A \times E} P\left((a,\varepsilon), \mathcal{A} \times \mathcal{E}\right) d\lambda \qquad \forall\ \mathcal{A} \times \mathcal{E} \in \mathbf{B}$$

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  - Savings = Investment: Find r such that $K(r) = A(r)$

Equilibrium

A **Recursive Competitive Equilibrium** is a:

(I) value function $v : S \to \mathbb{R}$ and policy functions $c : S \to \mathbb{R}$ and $a' : S \to \mathbb{R}$,

(II) firm's optimal capital and labor, $(K, L) \in \mathbb{R}^2_+$,

(III) prices $(r, w) \in \mathbb{R}^2_{++}$, and

(IV) a stationary probability measure $\lambda \in \Lambda(S, \mathbf{B})$

such that:

- Given prices (r, w), (c, a') solve consumer's problem with associated v
- Given prices (r, w), firm chooses (K, L) optimally
- Goods, Capital and Labor markets clear
- The distribution λ is a fixed point of T^* operator:

$$\lambda(\mathcal{A} \times \mathcal{E}) = \int_{A \times E} P\left((a, \varepsilon), \mathcal{A} \times \mathcal{E}\right) d\lambda$$

for all $(\mathcal{A} \times \mathcal{E}) \in \mathbf{B}$, given transition function P constructed from $a'(a, \varepsilon)$ and $\pi(\varepsilon' | \varepsilon)$.

Recursive Competitive Equilibrium

Existence of r^*

- Continuity:
  - Huggett (1993): Shows a stationary λ exists and $A(r)$ is continuous
  - Production: $K(r) = (\alpha/(r+\delta))^{1/(1-\alpha)}L$ is continuous wrt r

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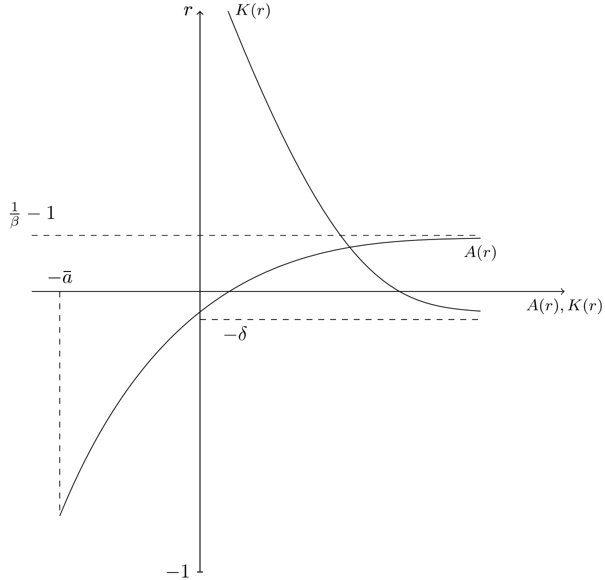
  - If $r = -1 < 0$ then savings generates zero returns, $A(-1) = 0$
  - If $1 + r \geq 1/\beta$ then savings diverges, $\lim_{r\to\frac{1}{\beta}-1} A(r) = \infty$

Recursive Competitive Equilibrium

Existence of r^*

- Limits: there exists r s.t. the Excess Demand function is:
  - Greater than zero:
$$\lim_{r \to -\delta} \Big(K(r) - A(r) \Big) = \infty$$
  - Less than zero:
$$\lim_{r \to (1/\beta - 1)} \Big(K(r) - A(r) \Big) = -\infty$$
- By continuity:
  - there exists a $r^* \in (-\delta, 1/\beta - 1)$ such that $K(r^*) - A(r^*) = 0$

Recursive Competitive Equilibrium



Today (2/26/18)

- Existence of equilibrium prices
- Computation of policy function and stationary distribution
- Precautionary Savings in General Equilibrium
- Applications:
  - Government Debt
  - Constrained Efficiency (see notes! cool stuff!)

Computation

Next Steps:

1. Calibration
2. Discretization of Markov processes
3. Agent's Problem: solving for policy functions
4. Aggregation: solving for the stationary distribution
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Calibration

Parameters: $(\beta, \sigma, \alpha, \delta, \rho, \sigma_\varepsilon, \underline{a})$

- $\alpha = 1/3$ corresponds to 2/3 labor share of US income
- $\delta = 0.06$ corresponds to estimates of capital depreciation
- $\rho = 0.95$ and $\sigma_\varepsilon = 0.2$ correspond to estimation of AR1 from PSID

$$\log(\varepsilon_t) = (1-\rho)\mu + \rho \log(\varepsilon_{t-1}) + \epsilon_t \quad \text{s.t. } \epsilon_t \sim \mathcal{N}(0, \sigma_\varepsilon^2)$$

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- Internally calibrated: $(\beta, \sigma, \underline{a})$
  - β corresponds to model-generated $K/Y \approx 3$
  - $\underline{a}$ corresponds to fraction of agents with negative wealth
  - σ corresponds to some statistic (e.g. median) of wealth distribution

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Note: A Shortcut

- Finding internally calibrated parameters $(\beta, \sigma, \underline{a})$ can be costly
- Alternatively, choose without Method of Moments

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- Find β^{CM} implied by complete markets:

$$1 = \beta^{CM} \left(\alpha K^{\alpha-1} L^{1-\alpha} + (1-\delta) \right)$$

$$\beta^{CM} = \frac{1}{1 + \alpha \frac{Y}{K} - \delta} = \frac{1}{1 + (1/3)(1/3) - 0.06} \approx 0.9514$$

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- $\underline{a} = 0$ for simplicity

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Discrete Markov Processes

Tauchen (1986)

- Denote $y \equiv \log(\varepsilon)$ and consider AR1:

$$y' = \rho y + \epsilon' \quad \text{s.t. } \epsilon \sim \mathcal{N}(0, \sigma_\epsilon^2)$$

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$$y_{n_\varepsilon} = 3 \cdot \frac{\sigma_\epsilon}{\sqrt{1 - \rho^2}}$$

$$y_1 = -y_{n_\varepsilon}$$

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$$= \text{Prob}\Big(y_j - \frac{1}{2}\Delta < \rho y_i + \epsilon \leq y_j + \frac{1}{2}\Delta\Big)$$

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&= \text{Prob}\Big(y_j - \rho y_i - \frac{1}{2}\Delta < \epsilon \leq y_j - \rho y_i + \frac{1}{2}\Delta\Big)
\end{aligned}$$

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&= \text{Prob}\Big(y_j - \frac{1}{2}\Delta < \rho y_i + \epsilon \leq y_j + \frac{1}{2}\Delta\Big) \\
&= \text{Prob}\Big(y_j - \rho y_i - \frac{1}{2}\Delta < \epsilon \leq y_j - \rho y_i + \frac{1}{2}\Delta\Big) \\
&= \Phi\left(\frac{y_j - \rho y_i + \frac{1}{2}\Delta}{\sigma_\epsilon}\right) - \Phi\left(\frac{y_j - \rho y_i - \frac{1}{2}\Delta}{\sigma_\epsilon}\right)
\end{aligned}
$$

where Φ is the Normal CDF since $\epsilon \sim \mathcal{N}(0, \sigma_\varepsilon^2)$

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- To find appropriate n_ε,
    - simulate $\{\hat{y}_t\}_{t=0}^T$
    - check serial correlation / variance against $(\rho, \sigma_\varepsilon^2)$
    - if not equal (up to the desired tolerance), then increase n_ε

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- To find appropriate n_ε,
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  - check serial correlation / variance against $(\rho, \sigma_\varepsilon^2)$
  - if not equal (up to the desired tolerance), then increase n_ε
- For highly persistent processes, other methods are more useful (c.f. Kopecky and Suen (2011))

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   - Preliminaries
   - Endogenous Gridpoint Method
   - Fixed Point Iteration
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Computational Preliminaries

Asset Grid

- Have already discretized $\varepsilon_i \equiv \exp(y_i)$
- Discretize asset grid with n_a nodes on $[\underline{a}, \overline{a}]$

$$a_i = \underline{a} + \left(\frac{i-1}{n_a - 1} \right)^{\theta} (\bar{a} - \underline{a}) \qquad \forall \; i = 1, \ldots, n_a$$

- General idea: want more nodes where policy functions least linear
- Let $\theta > 1$ to set more nodes near borrowing constraint (highest MPC)
- Denote the grids as $\mathbb{G}_a$ and $\mathbb{G}_\varepsilon$

Computational Preliminaries

Linear Interpolation

- Suppose a grid over X is $\mathbb{G}_x$
- Known functional values $f(x_i)$ for each point $x_i \in \mathbb{G}_x$
- Unknown functional values for $f(x)$ s.t. $x \notin \mathbb{G}_x$
- Linear interpolation: find i such that $x_i \leq \hat{x} \leq x_{i+1}$ and let

$$f(\hat{x}) = \left(1 - \frac{\hat{x} - x_i}{x_{i+1} - x_i}\right) f(x_i) + \frac{\hat{x} - x_i}{x_{i+1} - x_i} f(x_{i+1})$$

- Denote the interpolation of $\hat{x}$ onto a pair $x_i \in \mathbb{G}_x$ and $f(x_i)$ as:

$$f(\hat{x}) = \mathcal{I}\Big(x \in \mathbb{G}_x, f(x) \;\Big|\; \hat{x}\Big)$$

Computation

Next Steps:

1. Calibration
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   - Preliminaries
   - Endogenous Gridpoint Method
   - Fixed Point Iteration
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Conceptual Overview

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5. Update $a'(a, \varepsilon)$ using dampening (relaxation)
6. If $a'_{new}(a, \varepsilon) = a'_{old}(a, \varepsilon)$ up to the desired tolerance, stop.
   Otherwise, go back to Step 2 with $a'_{new}(a, \varepsilon)$

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where the last term is the savings choice in the next period, given by:

$$a''(a, \varepsilon, \varepsilon') = \mathcal{I}\left(a \in \mathbb{G}_a, a'_i(a, \varepsilon') \;\middle|\; a'_i(a, \varepsilon)\right)$$

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Impose $c'(a,\varepsilon,\varepsilon') \geq 0$ if violated.

Fixed Point Iteration

3. Compute an updated policy function for consumption from the Euler equation:

$$c_{i+1}(a, \varepsilon) = B_i(a, \varepsilon)^{-\frac{1}{\sigma}}$$

and impose $c_{i+1}(a, \varepsilon) \leq w\varepsilon + (1 + r)a - \underline{a}$ if violated.

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4. Compute an updated policy function for savings from the budget constraint:

$$\hat{a}'_{i+1}(a,\varepsilon) = w\varepsilon + (1+r) - c_{i+1}(a,\varepsilon)$$

and impose $\hat{a}'_{i+1}(a,\varepsilon) \geq \underline{a}$ if violated.

Fixed Point Iteration

5. Update the savings policy function by dampening with some fixed $\zeta \in (0,1)$:

$$a'_{i+1}(a,\varepsilon) = (1-\zeta)a'_i(a,\varepsilon) + \zeta \hat{a}'_{i+1}(a,\varepsilon)$$

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5. Update the savings policy function by dampening with some fixed $\zeta \in (0, 1)$:

$$a'_{i+1}(a, \varepsilon) = (1 - \zeta)a'_i(a, \varepsilon) + \zeta \hat{a}'_{i+1}(a, \varepsilon)$$

6. If $a'_{i+1} = a'_i$ up to the desired tolerance, then stop.
   More specifically: stop if the following holds for a fixed $\kappa \in \mathbb{N}_{++}$

$$\max_{(a,\varepsilon) \in \mathbb{G}_a \times \mathbb{G}_\varepsilon} \left| \frac{a'_{i+1}(a, \varepsilon) - a'_i(a, \varepsilon)}{1 + a'_i(a, \varepsilon)} \right| < 10^{-\kappa}$$

   Otherwise go back to step 2 with $a'_i \to a'_{i+1}$.

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Overview

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$$(1 - \omega_i)a_i + \omega_i a_{i+1} = a'(a, \varepsilon) \quad \Longrightarrow \quad \omega_i = \frac{a'(a, \varepsilon) - a_i}{a_{i+1} - a_i}$$

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$$(1-\omega_i)a_i + \omega_i a_{i+1} = a'(a,\varepsilon) \quad \implies \quad \omega_i = \frac{a'(a,\varepsilon) - a_i}{a_{i+1} - a_i}$$

- Iterate on the distribution using the T^* operator

$$\lambda_{i+1}(a',\varepsilon') = \int_{A\times E} P\left((a,\varepsilon),(a',\varepsilon')\right) d\lambda_i$$

Stationary Distribution

- See lecture notes for constructing weights (and Young (2010))
- Focus on iteration using T^* operator

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A. Update the distribution using the transition function for all $j = 1, \ldots, n_a^\lambda$, given $\lambda_i(a, \varepsilon)$:

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B. If $\lambda_{i+1} = \lambda_i$ up to the desired tolerance, then stop. Otherwise go back to step A with $\lambda_i \to \lambda_{i+1}$.

Computation

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Equilibrium Prices

Bisection

1. Set initial boundaries on the interest rate:

$$[\underline{r}_0, \bar{r}_0] = \left[-\delta + 0.01,\ \frac{1}{\beta} - 1 - 0.01\right]$$

   and set the initial interest rate to

$$r_0 = \frac{1}{2}\left(\underline{r}_0 + \bar{r}_0\right)$$

2. Given r_i, compute the savings policy function $a'(a, \varepsilon | r_i)$
3. Given r_i, compute the stationary distribution function $\lambda(a, \varepsilon | r_i)$ and corresponding density function $f(a, \varepsilon | r_i)$

Equilibrium Prices

4. Update the interest rate using bisection.
   - Compute excess demand:

$$E(r_i) = K(r_i) - A(r_i) = \left(\frac{\alpha}{r_i + \delta}\right)^{\frac{1}{1-\alpha}} L - \sum_{\varepsilon \in \mathbb{G}_\varepsilon} \sum_{a \in \mathbb{G}_a} a'(a, \varepsilon | r_i) f(a, \varepsilon | r_i)$$

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   - Compute $\hat{r} = \alpha A(r_i)^{\alpha-1} L^{1-\alpha} - \delta$
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   - Compute $\hat{r} = \alpha A(r_i)^{\alpha - 1} L^{1-\alpha} - \delta$
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5. If for fixed $\kappa \in \mathbb{N}_{++}$, $|r_{i+1} - r_i| < 10^{-\kappa}$ then stop;
   otherwise, go to step 2 with $r_i \to r_{i+1}$.

Today (2/27/18)

- Existence of equilibrium prices
- Computation of policy function and stationary distribution
- Precautionary Savings in General Equilibrium
- Applications:
  - Government Debt
  - Constrained Efficiency (lecture notes, check it out!)

Precautionary Savings

Aiyagari (1994)

- Case 1: $u(c) = \log(c)$ and $\rho = 0 \quad \rightarrow$

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    - Precautionary savings account for large fraction of aggregate output

$$\frac{K(r^*) - K(r^{cm})}{Y(r^*)} = 14\%$$

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  - When σ high, large smoothing motive
    (high utility cost of consumption fluctations)
  - When $\rho \rightarrow 1$, accumulate larger buffer with high shock
    (low shocks will be very persistent)

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- Case 3: CRRA with $\sigma = 2$ and $\rho = 0.9$

$$\frac{K(r^*) - K(r^{cm})}{Y(r^*)} = 5\%$$

Precautionary Savings

Comparative Statics

- Borrowing Constraint: Loosen to $\underline{a} < 0$
  - Less constrained: better able to smooth consumption
  - Precautionary Savings:

Precautionary Savings

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  - Asset Supply:
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    - increases in general equilibrium
  - Interest Rate: increases
    - Representative Firm demand assets
    - higher r encourages individuals to hold more assets

Precautionary Savings

Comparative Statics

- Risk Aversion: Increase in σ
  - Higher utility cost of consumption fluctuations
  - Precautionary Savings:

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Comparative Statics

- Risk Aversion: Increase in σ
    - Higher utility cost of consumption fluctuations
    - Precautionary Savings: increases
    - Asset Supply: increases in partial equilibrium (constant r)
    - Interest Rate: decreases
        - Excess supply over asset quantity Firm demands
        - lower r encourages individuals to hold fewer assets

Precautionary Savings

Comparative Statics

- Shock Volatility: Increase in σ_ε^2
  - Greater probability of low shock: binding borrowing constraint
  - Precautionary Savings: increases
  - Unconditional variance: $\sigma_\varepsilon^2(1-\rho^2)$
    - increase in ρ has same qualitative effect

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Government Debt

Aiyagari and McGrattan (1998)

- Q: What is the optimal level of public debt?
- Q: How do market incompleteness and idiosyncratic shocks affect it?
- A: Current post-war US debt (2/3 of GDP) close to optimal
- A: Debt effectively loosens individual borrowing constraints
- Include a government in previous model
  - Can issue public debt
  - Can tax income and lump-sum transfer resources to/from agents

Government Debt

Production

- Same as before!
- Representative Firm with $Y = zK^{\alpha}L^{1-\alpha}$
- Hires labor at wage w, rents capital at rental rate r
- Chooses (K, L) to set MP equal to prices

Government Debt

Consumer's Problem:

$$v(a,\varepsilon) = \max_{c,a',l} \frac{\left(c^{\eta} l^{1-\eta}\right)^{1-\sigma}}{1-\sigma} + \beta \sum_{\varepsilon' \in E} \pi(\varepsilon'|\varepsilon) v(a',\varepsilon')$$

$$\begin{aligned} \text{s.t.} \quad c + a' &\leq (1-\tau) w \varepsilon (1-l) + (1 + (1-\tau) r) a + T \\ a' &\geq \underline{a} \end{aligned}$$

Euler Equation:

$$u_c(c,l) \geq \beta (1 + (1-\tau) r) \sum_{\varepsilon' \in E} \pi(\varepsilon'|\varepsilon) u_c(c',l') \quad \text{w.e. if } a' > \underline{a}$$

Intratemporal Optimality:

$$(c/l) = \frac{\eta}{1-\eta} (1-\tau) w \varepsilon$$

Government Debt

Consumer's Problem:

$$v(a,\varepsilon) = \max_{c,a',l} \frac{\left(c^{\eta} l^{1-\eta}\right)^{1-\sigma}}{1-\sigma} + \beta \sum_{\varepsilon' \in E} \pi(\varepsilon'|\varepsilon) v(a',\varepsilon')$$

$$\text{s.t.} \quad c + a' \leq (1-\tau) w \varepsilon (1-l) + (1 + (1-\tau) r) a + T$$

$$a' \geq \underline{a}$$

Euler Equation:

$$u_c(c,l) \geq \beta(1 + (1-\tau) r) \sum_{\varepsilon' \in E} \pi(\varepsilon'|\varepsilon) u_c(c',l') \quad \text{w.e. if } a' > \underline{a}$$

Intratemporal Optimality:

$$(c/l) = \frac{\eta}{1-\eta} (1-\tau) w \varepsilon \quad \Longrightarrow \quad u_c(c,l) = \left[\eta \left(\frac{1-\eta}{\eta(1-\tau) w \varepsilon}\right)^{(1-\eta)(1-\sigma)}\right] c^{-\sigma}$$

Government Debt

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$$\begin{aligned} \text{s.t.} \quad c + a' &\leq (1-\tau) w \varepsilon (1-l) + (1 + (1-\tau) r) a + T \\ a' &\geq \underline{a} \end{aligned}$$

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Euler Equation:

$$h(\varepsilon)c^{-\sigma} \geq \beta(1+(1-\tau)r) \sum_{\varepsilon' \in E} \pi(\varepsilon'|\varepsilon) h(\varepsilon')(c')^{1-\sigma} \quad \text{w.e. if } a' > \underline{a}$$

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Government Debt

Government Budget Constraint

$$G + T + rB = (B' - B) + \tau(wL + rA)$$

- Consumes G each period (thrown into ocean)
- Issues lump-sum transfer T
- Services debt rB at interest rate r
- Issues new debt: $B' - B$
- Collects income taxes $\tau(wL + rA)$

Government Debt

Market Clearing

- Given distribution $\lambda(a, \varepsilon)$, labor market clearing:

$$L = \sum_{\varepsilon} \int_{\mathcal{A}} \varepsilon\big(1 - l(a, \varepsilon)\big)\lambda(da, \varepsilon)$$

Government Debt

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- Asset market clearing (demand from firms <u>and</u> gov't)

$$A = K + B$$

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Government Debt

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- Goods market clearing (by Walras' Law)

Calibration and Results

Calibration

- Mostly Standard
- Set $G/Y = 0.22$, $T/Y = 0.08$, $B/Y = 0.67$
- Set $\eta = 0.33$, $\sigma = 1.5$, $\rho = 0.6$, $\sigma_\varepsilon = 0.3$

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Results

- Choosing B/Y optimally, implies $B/Y \approx 2/3$
- US debt system ...
  "exactly balances the negative role of crowding out private capital and distorting labor supply and savings decisions through higher taxes" with the positive role of liquidity provision.

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Question:
What if $\rho \approx 1$ and $\sigma = 5$?

Thanks!