Lecture 3

(1) Precautionary Savings

(2) Impatience

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Last Time (2/6/18)

- **Permanent Income Hypothesis (PIH)**
  - Restrict asset space
  - Exogenously incomplete asset markets
  - Cannot write asset contracts contingent on specific future states

$$c_t + \frac{1}{1 + r_t} a_{t+1} \leq y_t + a_t$$

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- **"Strict" Permanent Income Hypothesis**
  - Utility: $u(c) = -(\alpha/2)(c_t - \bar{c})^2$
  - Time Preference: $\beta(1+r) = 1$
  - No Ponzi Condition

Last Time (2/6/18)

Empirical evaluation of theory

- Excess Sensitivity Puzzle:

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- Excess Smoothness Puzzle
  - Strict PIH predicts: $\sigma_{\Delta c} > \sigma_{\Delta y}$
  - In the data: $\sigma_{\Delta c} < \sigma_{\Delta y}$
  - Illustrated this puzzle using the Strict PIH model

Last Time (2/6/18)

Reconciliation of Puzzles: Campbell and Deaton (1989)

- Suppose individuals have more information than econometrician

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- Classic source of bias: estimating income variability with error

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Reconciliation of Puzzles: Campbell and Deaton (1989)

- Suppose individuals have more information than econometrician
- Classic source of bias: estimating income variability with error
- **Strategy:** Use information on savings as predictor of income growth

Last Time (2/6/18)

- Assume:

$$\begin{pmatrix} \Delta y_t \\ \frac{1}{1+r}\Delta a_{t+1} \end{pmatrix} = \begin{pmatrix} \zeta_{11} & \zeta_{12} \\ \zeta_{21} & \zeta_{22} \end{pmatrix} \begin{pmatrix} \Delta y_{t-1} \\ \frac{1}{1+r}\Delta a_t \end{pmatrix} + \begin{pmatrix} u_{1t} \\ u_{2t} \end{pmatrix}$$

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- Implies parameter restrictions:

$$\zeta_{21} = \zeta_{11}$$

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- Suppose $\zeta_{21} = \zeta_{11} - \chi$ such that:

$$A = \begin{pmatrix} \zeta_{11} & \zeta_{12} \\ \zeta_{11} - \chi & \zeta_{12} + 1 + r \end{pmatrix}$$

Last Time (2/6/18)

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Last Time (2/6/18)

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- Cases: $\chi \to 0$, $\chi \to 1 + r$
- Magnitude of excess sensitivity determines consumption response
- "There is no contradiction between excess sensitivity and excess smoothness; they are the same phenomenon."

Agenda

- Other mechanisms that generate excess sensitivity and smoothness?

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  - Prudence in utility
- Friedman / Buffer Stock model
  - Add impatience to precautionary motives
  - Marginal Propensity to Consume out of income shocks

Today

- **Start: Buffer Stock Savings Model**
  - Borrowing Constraints
  - Prudence
  - Patience

Precautionary Savings

Borrowing Constraints

- Empirical Observation (Parker (1999), Souleles (1999), others)

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- Consumption is sensitive to *anticipated* income changes
- Study government transfers (tax rebates, social security changes)
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- Suggestive of borrowing constraints impeding consumption smoothing

Precautionary Savings

Borrowing Constraints

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Precautionary Savings

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- Add *No Borrowing Constraint*: $a_{t+1} \geq 0$
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- Optimal consumption is:

$$c_t = \left\{ \begin{array}{ll} \mathbb{E}_{t-1}[c_t] & \text{if } a_{t+1} > 0 \\ y_t + a_t & \text{if } a_{t+1} = 0 \end{array} \right\}$$

Precautionary Savings

Borrowing Constraints

- Recall optimal savings dynamics:

$$\frac{1}{1+r}\Delta a_{t+1} = -\sum_{j=1}^{\infty}\left(\frac{1}{1+r}\right)^{j}\mathbb{E}_t[\Delta y_{t+j}]$$

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- Then $\Delta a_{t+1} = 0$
- All innovations to income are consumed

Precautionary Savings

Borrowing Constraints

- Suppose iid shocks: $y_t = \epsilon_t$ such that $\epsilon \sim \mathcal{N}(0, \sigma_\epsilon^2)$

Precautionary Savings

Borrowing Constraints

- Suppose iid shocks: $y_t = \epsilon_t$ such that $\epsilon \sim \mathcal{N}(0, \sigma_\epsilon^2)$
- Then: $\Delta y_{t+j} = \epsilon_{t+j} - \epsilon_{t-1+j}$

Precautionary Savings

Borrowing Constraints

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- Optimal savings dynamics:

$$\frac{1}{1+r}\Delta a_{t+1} = -\sum_{j=1}^{\infty}\left(\frac{1}{1+r}\right)^j \mathbb{E}_t[\Delta y_{t+j}] = \frac{1}{r}\epsilon_t$$

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Precautionary Savings

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- Savings is a random walk
- Borrowing constraint binds with probability one as $t \to \infty$

Precautionary Savings

Borrowing Constraints

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Precautionary Savings

Borrowing Constraints

- Suppose iid shocks: $y_t = \epsilon_t$ such that $\epsilon \sim \mathcal{N}(0, \sigma_\epsilon^2)$
- Optimal consumption:

$$c_t = \left\{ \begin{array}{cc} \frac{r}{1+r}(a_t + y_t) & \text{if } a_{t+1} > 0 \\ y_t + a_t & \text{if } a_{t+1} = 0 \end{array} \right\} = \min\left\{ y_t + a_t,\ \frac{r}{1+r}(a_t + y_t) \right\}$$

such that

$$c_t = y_t + a_t - \underbrace{\frac{1}{1+r}a_{t+1}}_{=0} \quad \text{or} \quad c_t \leq y_t + a_t$$

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- Agent is constrained when total income is negative:

$$\frac{r}{1+r}(a_t + y_t) > y_t + a_t \quad \Rightarrow \quad y_t < -a_t$$

Precautionary Savings

Borrowing Constraints

- Suppose iid shocks: $y_t = \epsilon_t$ such that $\epsilon \sim \mathcal{N}(0, \sigma_\epsilon^2)$
- Rewrite optimal consumption:

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Precautionary Savings

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- Future borrowing constraints affect current consumption:

$$\text{if } a_{t+1} > 0, \quad c_t = \mathbb{E}_t[\min\{y_{t+1} + a_{t+1}, \mathbb{E}_{t+1}[c_{t+2}]\}]$$

Precautionary Savings

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- What happens if σ_ϵ increases?

Precautionary Savings

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- What happens if σ_ϵ increases? $\rightarrow$ Increase Precautionary Savings

Today

- **Buffer Stock Savings Model**
  - Borrowing Constraints
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Precautionary Savings

Prudence

- Another mechanism: Prudence (Kimball, 1990)

Precautionary Savings

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- Preferences for smooth consumption profile

  $\Rightarrow$

  Self-insurance against low income realizations

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  Income variability increases Precautionary Motive

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  Income variability increases Precautionary Motive
- Keep in mind: CRRA but NOT Quadratic or CARA

$$u(c) = \frac{c^{1-\sigma}}{1-\sigma}$$

Precautionary Savings

Prudence

- Consider: $u : \mathbb{R}_+ \to \mathbb{R}$, $u'(c) > 0$, $u''(c) < 0$

Precautionary Savings

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- Consider: $u : \mathbb{R}_+ \to \mathbb{R}$, $u'(c) > 0$, $u''(c) < 0$
- Arrow-Pratt measure of of absolute risk aversion:

$$A(c) \equiv -\frac{u''(c)}{u'(c)}$$

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- Decreasing absolute risk aversion (DARA) implies:

$$A'(c) = -\frac{u'''(c)}{u'(c)} + \left(\frac{u''(c)}{u'(c)}\right)^2 < 0$$

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- Condition on the third derivative of the utility function:

$$u'''(c) > \frac{u''(c)^2}{u'(c)} > 0$$

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- *Prudence*: Convexity of the marginal utility function, e.g. $u'''(c) > 0$

Precautionary Savings

Examples

- Keep in mind: CRRA but not Quadratic or CARA
- Constant Relative Risk Aversion Utility

$$u(c) = \frac{c^{1-\sigma}}{1-\sigma}$$

- Constant Absolute Risk Aversion Utility

$$u(c) = -e^{-\alpha c}$$

Precautionary Savings

Prudence

- Prudence implies:
  - Uncertainty increases
  - Consumption today decreases
  - Savings for tomorrow increases

Precautionary Savings

Prudence

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- Consider two period example (Leland, 1968)
  - Board work
  - T period example

Next Steps

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  - Borrowing Constraints
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  - Patience

Patience

- Recall canonical consumption-savings problem:

$$v(a_t, y_t) = \max_{c_t, a_{t+1}} \; u(c_t) + \beta \mathbb{E}_t \left[v(a_{t+1}, y_{t+1}) \right]$$

$$\begin{aligned} \text{s.t.} \quad c_t + a_{t+1} &\leq y_t + (1+r)a_t \\ a_{t+1} &\geq \underline{a} \end{aligned}$$

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- Euler equation:

$$u'(c_t) \geq \beta(1+r)\mathbb{E}_t[u'(c_{t+1})] \qquad \text{w.e. if } a_{t+1} > \underline{a}$$

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  - Patient: $\beta(1+r) \geq 1$

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$$a_{t+1} \geq \underline{a}$$

- Euler equation:

$$u'(c_t) \geq \beta(1+r) \mathbb{E}_t[u'(c_{t+1})] \qquad \text{w.e. if } a_{t+1} > \underline{a}$$

- $\beta(1+r)$ determines incentives to intertemporally substitute consumption
  - Patient: $\beta(1+r) \geq 1$
  - Impatient: $\beta(1+r) < 1$

Patience

Next Steps

- How far can we go on theoretical characterization?
- What can we say about convergence?
- When does there exist some endogenous upper bound $\bar{a}$?

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    agents run down savings when a sufficiently high
  - If there does not exist such an upper bound $\bar{a}$,
    savings diverges
- Four Cases:
  (I) Deterministic vs Stochastic Income
  (II) $\beta(1+r) \geq 1$ vs $\beta(1+r) < 1$

Patience

- Start with Deterministic Income
  - $\beta(1+r) > 1$, $\beta(1+r) = 1$ and $\beta(1+r) < 1$
  - Show that $\beta(1+r) \leq 1$ converges

Patience

- Start with Deterministic Income
  - $\beta(1+r) > 1$, $\beta(1+r) = 1$ and $\beta(1+r) < 1$
  - Show that $\beta(1+r) \leq 1$ converges
- Then consider Stochastic Income
  - $\beta(1+r) > 1$, $\beta(1+r) = 1$ and $\beta(1+r) < 1$
  - Show that $\beta(1+r) < 1$ converges

Patience

Deterministic, $\beta(1+r) > 1$

- For simplicity, consider $y_t = \bar{y} < \infty$ for all t

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- Since utility is concave, $u''(c) < 0$, then:

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- Incentives to save too large, indefinitely postpones consumption:

$$\lim_{t\to\infty} c_t = \infty$$

Patience

Deterministic $y_t = \bar{y}$ **and** $\beta(1+r) = 1$

- Case 1: $a_{t+1} > \underline{a}$ for all t
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Patience

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Patience

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Patience

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- Case 2: $y_t = \bar{y}$ sufficiently low such that $a_{t+1} = \underline{a}$ for some t
- We can find a date s.t. human wealth reaches a maximum

$$\tau \equiv \arg\sup_t \left\{ \frac{r}{1+r} \sum_{j=0}^{\infty} \left(\frac{1}{1+r} \right)^j y_{t+j} \right\}$$

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  -
  -
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- For $t < \tau$, $c_{t+1} > c_t$
  - due to borrowing constraints
  - wants to borrow against future income
- For $t \geq \tau$, $c_{t+1} = c_t$
  - agent will have accumulated sufficiently high stock of savings
  - human wealth is decreasing, no incentive to borrow

Patience

Deterministic $y_t = \bar{y}$ and $\beta(1+r) < 1$

- For simplicity, let $\underline{a} = 0$ (No Borrowing Constraint)

Patience

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- Iterate forward on Budget Constraint:

$$c_t + 0 = \bar{y} + (1+r)a_t$$

$$c_{t+1} + 0 = \bar{y} + 0$$

$$c_{t+2} + 0 = \bar{y} + 0$$

$$\vdots \quad = \quad \vdots$$

- Then for all $j > 0$,

$$\boxed{a_{t+1+j} = 0 \quad \text{and} \quad c_{t+j} = \bar{y}}$$

Patience

Deterministic $y_t = \bar{y}$ **and** $\beta(1+r) < 1$

- **Case 2:** at time t, $a_{t+1} > 0$

Patience

Deterministic $y_t = \bar{y}$ **and** $\beta(1+r) < 1$

- **Case 2:** at time t, $a_{t+1} > 0$
- But first...

Patience

Useful Aside: cash-in-hand

- Rewrite consumption-savings problem with new state variable

$$x_t \equiv y_t + (1+r)a_t$$

Patience

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$$x_{t+1} = (1+r)(x_t - c_t) + y_{t+1}$$

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- Consumption Savings Problem:

$$v(x_t) = \max_{c_t \leq x_t} u(c_t) + \beta \mathbb{E}[v(x_{t+1})]$$

$$\text{s.t.} \quad x_{t+1} = (1+r)(x_t - c_t) + y_{t+1}$$

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- Euler equation (does not change):

$$u'(c_t) \geq \beta(1+r)v'(x_{t+1})$$

Patience

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Patience

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$$\bar{y} > \bar{y} \quad \rightarrow\leftarrow \qquad \text{[Concavity]}$$

Taking Stock

Deterministic $y_t = \bar{y}$

- If $\beta(1+r) > 1$, then $c_t \to \infty$ as $t \to \infty$
- If $\beta(1+r) = 1$, then $\exists\ \tau$ such that $c_{t+1} = c_t$ for all $t \geq \tau$
- If $\beta(1+r) < 1$, then $c_t = \bar{y}$ and $a_{t+1} = 0$ for all t sufficiently large

Next: Stochastic Case.

Patience

Stochastic Case

- Useful theorem: **Doob's Martingale Convergence Theorem (MCT)**
- Define:

$$M_t \equiv \big(\beta(1+r)\big)^t u'(c_t)$$

- Rewrite Euler equation:

$$\big(\beta(1+r)\big)^t u'(c_t) \geq \big(\beta(1+r)\big)^{t+1} \mathbb{E}_t[u'(c_{t+1})]$$

$$M_t \geq \mathbb{E}_t[M_{t+1}]$$

- If $M_t > 0$ for all t and follows a super martingale, then M_t converges almost surely to a finite limit

$$\lim_{t\to\infty} M_t = \bar{M} < \infty$$