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Advanced Topics in Macroeconomics:
Quantitative Macroeconomics

Lecture Notes[1]

Contents

[1]These lecture notes are incomplete and may contain mistakes, please use with caution. Please send comments to erick.sager@gmail.com. I thank Chris Carroll and Fabrizio Perri for their lecture notes and teachings. This set of lecture notes draws heavily from their work and notes.

1 Aggregation Results

1.1 Representative Agents

Representative Firms: Consider an economy with N^f firms, indexed by $i = 1, \ldots, N^f$. Each firm produces a homogeneous consumption good using the same production function $y^i = zF(k^i, n^i)$, where z is a (common) aggregate productivity, k^i is firm i's capital input and n^i is firm i's labor input. Assume that the function F is strictly increasing, strictly concave, twice continuously differentiable in both arguments and exhibits constant returns to scale.

Suppose that each firm purchases factor inputs in a perfectly competitive market, and therefore the firms are price takers. Then firm i chooses capital and labor inputs to maximize static profits:

$$\pi^i = \max_{k^i, n^i} \; zF(k^i, n^i) - wn^i - (r + \delta)k^i$$

with optimality conditions:

$$r = zF_k(k^i, n^i) - \delta$$

$$w = zF_n(k^i, n^i)$$

Because F has constant returns to scale, its first derivatives F_k and F_n are homogeneous of degree zero. Therefore we can rewrite the functions as $f_k(k^i/n^i) \equiv F_k(k^i/n^i, 1)$ and $f_n(k^i/n^i) \equiv F_n(k^i/n^i, 1)$, or:

$$\frac{F_k(k^i, n^i)}{F_n(k^i, n^i)} = \frac{f_k(k^i/n^i)}{f_n(k^i/n^i)} = \frac{r + \delta}{w}$$

By strict concavity of F and constant returns to scale, we know that f_k/f_n is strictly decreasing in k^i/n^i. Strict monotonicity implies that the function $g(k^i/n^i) \equiv f_k(k^i/n^i)/f_n(k^i/n^i)$ is invertible, so that:

$$\frac{k^i}{n^i} = g^{-1}\left(\frac{r + \delta}{w}\right) \quad \forall \; i = 1, 2, \ldots, N^f$$

But since the right hand side of the equation does not depend on a specific firm i, this relationship holds for each $i = 1, \ldots, N^f$. Therefore, denote average capital and labor inputs as K and L respectively and write:

$$\frac{k^i}{n^i} = \frac{K}{N} \quad \forall\ i = 1, 2, \ldots, N^f$$

Proposition 1 (Euler's Theorem)
Let $f : \mathbb{R}^n_+ \to \mathbb{R}$ be continuously differentiable on $\mathbb{R}^n_{++}$. Then f is homogeneous of degree k if and only if for all $x \in \mathbb{R}^n_{++}$ we have

$$kf(x) = \sum_{i=1}^{n} \frac{\partial f(x)}{\partial x_i} \cdot x_i$$

Proof. See Kim Border's website (click here) for full proof. □

The gist of the result is as follows. Suppose that we aggregate across all firms' output, then we can construct output by using properties of the constant returns to technology production function:

$$\begin{aligned}
\sum_{i=1}^{N^f} zF(k^i, n^i) &= z \sum_{i=1}^{N^f} \left(F_k(k^i, n^i)k^i + F_n(k^i, n^i)n^i \right) \\
&= z \sum_{i=1}^{N^f} \left(f_k(K/N)k^i + f_n(K/N)n^i \right) \\
&= z f_k(K/N) \left(\sum_{i=1}^{N^f} k^i \right) + z f_n(K/N) \left(\sum_{i=1}^{N^f} n^i \right) \\
&= z f_k(K/N)K + z f_n(K/N)N \\
&= zF(K, N)
\end{aligned}$$

To build intuition, consider the example $F(k, n) = k^\alpha n^{1-\alpha}$ for $\alpha < 1$.

Representative Household: Suppose that there are N^h households, indexed by $i = 1, 2, \ldots, N^h$. Each household has the same preference ordering over streams of consumption $c^i \equiv \{c^i_t\}_{t=0}^{\infty}$ given by:

$$U(c^i) = \sum_{t=0}^{\infty} \beta^t u(c^i_t)$$

where $u : \mathbb{R}_+ \to \mathbb{R}$ is a strictly increasing and strictly concave. Each household is endowed with an identical amount of capital, $k^i_0 = k_0$ for all $i = 1, 2, \ldots, N^h$. Lastly, each household is a price taker in competitive markets.

To reiterate, each household faces the same prices, has the same endowment and have the same *strictly concave* preferences over consumption. Therefore, each household will make the same consumption decisions and can be treated as a representative household. Critically, the utility function must be strictly concave. We will justify this claim in Section 1.3

1.2 Gorman Aggregation

Suppose there are N consumers, indexed by $i = 1, 2 \ldots, N$, and M consumption goods, denoted $c \equiv \{c_1, \ldots, c_M\}$. Consumers value consumption according to strictly increasing and concave utility functions given by $u_i : \mathbb{R}^M \to \mathbb{R}$. Assume that consumers are price takers facing prices $p \equiv \{p_1, \ldots, p_M\}$ and are endowed with $w \equiv \{w_i\}_{i=1}^N$ units of wealth.

Suppose that agents with wealth w_i that face prices p choose a vector of consumption according to the decision rule $c^i(p, w_i)$, where $c^i : \mathbb{R}^M \times \mathbb{R} \to \mathbb{R}^M$. Aggregate demand for the vector of M goods is:

$$C(p, w) = \sum_{i=1}^{N} c^i(p, w_i)$$

Clearly, aggregate demand depends on the *distribution of wealth* across individual agents. Under what conditions on primitives can we write aggregate demand as a function of aggregate wealth $W \equiv \sum_{i=1}^N w_i$ instead of as a function of the wealth distribution w?

In order for the distribution not to matter for aggregate wealth, it must be the case that aggregate demand W does not change in response to any arbitrary redistribution in wealth between the N agents. That is, if we take two arbitrary wealth distributions w and w' for which aggregate wealth is equal ($W = W'$), then aggregate demand must be equal $C(p, w) = C(p, w')$.

Assume that from some wealth distribution w we define an arbitrary redistribution as $dw = (dw_1, \ldots, dw_N)$ such that $\sum_{i=1}^N dw_i = 0$. Also assume that consumption decision rules are differentiable. There is no total change in consumption due to the redistribution,

$$\sum_{i=1}^{N} \frac{\partial c_j^i(p, w_i)}{\partial w_i} dw_i = 0 \quad \forall\, j = 1, 2, \ldots, M$$

if each consumer $i = 1, \ldots, N$ changes his consumption in the same way in response to a change in wealth,

$$\frac{\partial c_j^i(p, w_i)}{\partial w_i} = \frac{\partial c_j^k(p, w_k)}{\partial w_k} \quad \forall\ i, k = 1, \ldots, N\ \forall j = 1, \ldots, M \tag{1}$$

The above equation (1) holds if all agents have the same marginal propensity to consume (MPC) out of wealth, a sufficient condition for which is an identical and linear Engel curve:

$$c^i(p, w_i) = a^i(p) + b(p)w_i$$

If each agent has the same, linear Engel curve, then aggregate demand is a function of aggregate wealth:

$$C(p, w) = \sum_{i=1}^{N} c^i(p, w_i)$$

$$C(p, w) = \sum_{i=1}^{N} \left(a^i(p) + b(p)w_i\right)$$

$$C(p, W) = \left(\sum_{i=1}^{N} a^i(p)\right) + b(p)W$$

The general result is now stated.

Proposition 2 (Gorman Form)
Aggregate consumption can be expressed as a function of aggregate wealth if and only if agents have preferences that admit indirect utility functions with the Gorman form and identical coefficients on wealth across agents:

$$v_i(p, w_i) = a^i(p) + b(p)w_i$$

Proof. See Gorman (1953) and a discussion in Deaton and Muellbauer (1980). □

The result shows how to construct the preferences of a representative consumer by aggregating the preferences of individual agents. The result does not require any assumptions on technologies or asset markets. However, the result does require strong restrictions on the class of utility functions.

Simple Example: To build intuition, here is an example out of intermediate micro. Suppose there are two consumption goods $(M = 2)$ and agents possess quasilinear utility functions of the form $u_i(c_1, c_2) = \alpha c_1 + v_i(c_2)$. Suppose that agents are endowed with wealth $\{w_i\}_{i=1}^N$ and face goods prices (p_1, p_2).

Consider the choice problem of some agent i:

$$U_i(p_1, p_2, w_i) = \max_{c_1, c_2 \geq 0} \left\{ \alpha c_1 + v_i(c_2) \quad \text{s.t.} \quad p_1 c_1 + p_2 c_2 \leq w_i \right\}$$

First order conditions at an interior solution are:

$$\alpha = p_1 \lambda_i$$

$$v_i'(c_2) = p_2 \lambda_i$$

where λ_i is the multiplier on the budget constraint. Therefore the optimal allocation and indirect utility function are:

$$c_1 = \frac{w_i}{p_1} - \frac{p_2}{p_1} v_i'^{-1}\left(\alpha \frac{p_2}{p_1}\right)$$

$$c_2 = v_i'^{-1}\left(\alpha \frac{p_2}{p_1}\right)$$

$$U_i(p_1, p_2, w_i) = \left(\alpha \frac{w_i}{p_1} - \frac{p_2}{p_1} v_i'^{-1}\left(\alpha \frac{p_2}{p_1}\right)\right) + v_i\left(v_i'^{-1}\left(\alpha \frac{p_2}{p_1}\right)\right)$$

First notice that the optimal consumption of both goods take the general form $c_j(p, w) = a_j(p) + b_j(p) w$ with $b_1(p) = 1/p_1$ and $b_2(p) = 0$. Next notice that the indirect utility function can be written in Gorman form:

$$U_i(p_1, p_2, w_i) = \left[v_i\left(v_i'^{-1}\left(\alpha \frac{p_2}{p_1}\right)\right) - \frac{p_2}{p_1} v_i'^{-1}\left(\alpha \frac{p_2}{p_1}\right) \right] + \frac{\alpha}{p_1} w_i$$

Another Example: To build intuition, here is another easy example. Suppose now that agents have CES preferences with elasticity of substitution σ. Suppose an agent has wealth w and faces prices (p_1, p_2). The agent's choice problem is then:

$$U(p_1, p_2, w) = \max_{c_1, c_2 \geq 0} \left\{ \left(c_1^{1-\frac{1}{\sigma}} + c_2^{1-\frac{1}{\sigma}} \right)^{\frac{\sigma}{\sigma-1}} \quad \text{s.t.} \quad p_1 c_1 + p_2 c_2 \leq w \right\}$$

Denote the multiplier on the budget constraint by λ. For notational convenience, denote:

$$C = \left(c_1^{1-\frac{1}{\sigma}} + c_2^{1-\frac{1}{\sigma}}\right)^{\frac{\sigma}{\sigma-1}}$$

First order conditions for good $i = 1, 2$ are:

$$C^{\frac{1}{\sigma}} c_i^{-\frac{1}{\sigma}} = \lambda p_i$$

Rearranging:

$$C^{\frac{1}{\sigma}} c_i^{-\frac{1}{\sigma}} = \lambda p_i$$

$$c_i = C(\lambda p_i)^{-\sigma}$$

$$\left(\sum_{i=1}^{s} c_i^{1-\frac{1}{\sigma}}\right)^{\frac{\sigma}{\sigma-1}} = C\lambda^{-\sigma}\left(\sum_{i=1}^{s} p_i^{1-\sigma}\right)^{\frac{\sigma}{\sigma-1}}$$

$$\lambda^{-1} = \left(\sum_{i=1}^{s} p_i^{1-\sigma}\right)^{\frac{1}{1-\sigma}}$$

For convenience, denote $P \equiv \lambda^{-1}$. Substituting λ back into the FOC:

$$c_i = \left(\frac{p_i}{P}\right)^{-\sigma} C$$

Substitute the policy function into the budget constraint:

$$\sum_{i=1}^{2} p_i \left(\frac{p_i}{P}\right)^{-\sigma} C = w$$

$$\sum_{i=1}^{2} (p_i)^{1-\sigma} P^{\sigma} C = w$$

$$PC = w$$

Substituting $PC = w$ back into the policy function we obtain:

$$c_i = \left(\frac{p_i}{P}\right)^{-\sigma} P^{-1} w$$

Therefore the policy function is linear in wealth. The indirect utility function also takes the

Gorman form:

$$C = P^{-1}w$$

1.3 Aggregation with Complete Markets

Negishi Approach: In economies that do not admit Gorman aggregation, we may still recover a representative consumer under alternative assumptions on primitives. Negishi (1960) proposes a method to compute the competitive equilibrium prices and allocations of complete markets economies with heterogeneous households for which the welfare theorems hold.

If the First Welfare Theorem holds, then any competitive equilibrium is Pareto efficient. Therefore a competitive equilibrium of an economy with heterogeneous agents can be found as a solution to the Social Planner's Problem with the correctly chosen Pareto weights assigned to each agent. The correct Pareto weights are those that recover the competitive equilibrium allocation of the decentralized economy.

To fix ideas, consider a competitive economy and we will then apply Negishi's method to show that the competitive allocation is a decentralization of a particuar Social Planner's Problem.

Competitive Allocation: Consider an infinite horizon growth economy. There are two types of agents, consumers and firms. Suppose there are two $(N = 2)$ consumers. Consumers are endowed with identical preferences $u : \mathbb{R}_+ \to \mathbb{R}$ that are strictly increasing, strictly concave, twice continuously differentiable and satisfy Inada conditions. Consumers are endowed with initial wealth $\{a_0^1, a_0^2\}$ and face a sequence of given prices $p \equiv \{p_t\}_{t=0}^{\infty}$. Consumers trade assets in complete markets, so that there are no constraints on intertemporal transfers (e.g. no borrowing constraints on debt). Therefore the consumer's problem is:

$$v(p, a_0^i) = \max_{\{c_t^i\}_{t=0}^{\infty}} \left\{ \sum_{t=0}^{\infty} \beta^t u(c_t^i) \quad \text{s.t.} \quad \sum_{t=0}^{\infty} p_t c_t^i \leq p_0 a_0^i \right\}$$

Let λ^i be the multiplier on the budget constraint for agent $i \in \{1, 2\}$. First order conditions yield:

$$\beta^t u'(c_t^i) = \lambda^i p_t$$

which implies:

$$\frac{u'(c_t^1)}{u'(c_t^2)} = \frac{\lambda^1}{\lambda^2} \tag{2}$$

Turning to firms, suppose there is a representative firm that owns physical capital and makes investment decisions. The firm is endowed with k_0 units of capital at time-0. The household owns the shares in the firm and therefore the firm uses the consumers' wealth to finance investments. In mathematical notation, denote the time-0 value of the firm by A_0, and note that $a_0^1 + a_0^2 = A_0$. Therefore, the present value of the firm's profits is:

$$A_0 = \max_{\{k_{t+1}\}_{t=0}^{\infty}} \left\{ \sum_{t=0}^{\infty} \left(\frac{p_t}{p_0} \right) \left(f(k_t) + (1-\delta)k_t - k_{t+1} \right) \right\}$$

where the production function $f : \mathbb{R}_+ \to \mathbb{R}_+$ is strictly increasing, strictly concave and differentiable. The firm's first order equation is:

$$1 = \frac{p_{t+1}}{p_t} \Big(f'(k_{t+1}) + 1 - \delta \Big)$$

Notice that the firm's value can be written recursively as well. Suppose k_t is given and k_{t+1} is the optimal allocation, then:

$$A_t = \Big(f(k_t) + (1-\delta)k_t - k_{t+1} \Big) + \frac{p_{t+1}}{p_t} A_{t+1}$$

Lastly, the period-by-period resource constraint in this economy can be obtained from the consumers' budget constraints:

$$p_t c_t^i + \sum_{\tau > t} p_\tau c_\tau^i = p_t a_t^i$$

$$p_t c_t^i + p_{t+1} a_{t+1}^i = p_t a_t^i$$

$$c_t^i + \frac{p_{t+1}}{p_t} a_{t+1}^i = a_t^i$$

$$\sum_{i=1}^{2} c_t^i + \frac{p_{t+1}}{p_t} \sum_{i=1}^{2} a_{t+1}^i = \sum_{i=1}^{2} a_t^i$$

$$C_t + \frac{p_{t+1}}{p_t} A_{t+1} = A_t$$

$$C_t + k_{t+1} = f(k_t) + (1-\delta)k_t$$

Negishi Planner: Now consider the Negishi Planner's problem of choosing allocations to maximize social welfare subject to the resource constraint:

$$v^{NP}(\mu_1, \mu_2, k_0) = \max_{\{c_t^1, c_t^2, k_{t+1}\}_{t=0}^{\infty}} \left\{ \sum_{t=0}^{\infty} \beta^t \Big[\mu_1 u(c_t^1) + \mu_2 u(c_t^2) \Big] \quad \text{s.t.} \quad c_t^1 + c_t^2 + k_{t+1} = f(k_t) + (1-\delta) k_t \right\}$$

where k_0 is given and (μ_1, μ_2) are the planner's weights for each consumer.

Let $\beta^t \lambda_t^{RC}$ be the multiplier on the resource constraint. First order conditions are for all t:

$$\mu_i u'(c_t^i) = \lambda_t^{RC} \quad \forall\ i = 1, 2$$

$$\lambda_t^{RC} = \beta \lambda_{t+1}^{RC} \Big(f'(k_{t+1}) + 1 - \delta \Big)$$

which implies for all t:

$$\frac{u'(c_t^1)}{u'(c_t^2)} = \frac{\mu_2}{\mu_1} \tag{3}$$

Equation 3 tells us that (i) the planner fully insures each consumer by keeping their relative marginal utilities of consumption constant, and (ii) consumption is allocated to each consumer proportionately to its planner weights.

Implementing the Planner's Allocation: We may now selection the Negishi weights so that the planner implements the competitive equilibrium allocation. In particular we wish to choose weights so that the consumers' and firm's optimality conditions in competitive equilibrium are equivalent to those in the Planner's Problem. Combining the risk sharing condition under the competitive equilibrium allocation (equation 2) and under the Negishi Planner's allocation (equation 3) we obtain a condition on the Negishi weights:

$$\frac{\mu_2}{\mu_1} = \frac{\lambda_1}{\lambda_2}$$

Given the same initial capital, these Negishi weights will also ensure that competitive equilibrium prices induce the same path for investment under both competitive and centralized allocations. This follows from a straight forward comparison of the competitive firm's first order condition and the Negishi planner's intertemporal optimality condition:

$$p_t = \beta p_{t+1} \Big(f'(k_{t+1}) + 1 - \delta \Big)$$

$$\lambda_t^{RC} = \beta \lambda_{t+1}^{RC} \Big(f'(k_{t+1}) + 1 - \delta \Big)$$

Therefore the competitive equilibrium prices can be obtained from the sequence of multipliers on the Negishi planner's resource constraint:

$$\frac{\lambda_t^{RC}}{\lambda_{t+1}^{RC}} = \frac{p_t}{p_{t+1}}$$

Intuitively, the multiplier on the resource constraint is the marginal value of an extra unit of consumption. In competitive equilibrium, the price is also the marginal value of an additional unit of consumption.

Constantinides Aggregation: Constantinides (1982) provides a generalization of the Negishi Planner Problem. Consider the following Negishi Planner Problem:

$$\max_{\{\{c_t^i\}_{i=1}^N, k_{t+1}\}_{t=0}^\infty} \left\{ \sum_{t=0}^{\infty} \beta^t \sum_{i=1}^{N} \mu_i u(c_t^i) \quad \text{s.t.} \quad \sum_{i=1}^{N} \pi_i c_t^i + k_{t+1} = f(k_t) + (1-\delta)k_t \right\}$$

which is the same problem as above, except now N is an arbitrary number and there is a measure π_i of each type of household i such that $\sum_{i=1}^N \pi_i = 1$.

Constantinides (1982) demonstrated how to decompose / split the Negishi Planner's Problem and proved equivalence. For the first component, solve the static problem of allocating consumption to each consumer $i = 1, \ldots, N$ given a value of aggregate consumption C_t:

$$U(C_t) = \max_{\{c_t^i\}_{i=1}^N} \left\{ \sum_{i=1}^{N} \mu_i u(c_t^i) \quad \text{s.t.} \quad \sum_{i=1}^{N} \pi_i c_t^i \leq C_t \right\}$$

Given the solution to the first component, we can choose the level of aggregate consumption in the second component problem:

$$\max_{\{C_t, k_{t+1}\}_{t=0}^\infty} \left\{ \sum_{t=0}^{\infty} \beta^t U(C_t) \quad \text{s.t.} \quad C_t + k_{t+1} = f(k_t) + (1-\delta)k_t, \; k_0 \text{ given} \right\}$$

This second component problem is a representative agent problem and allows us to obtain a solution for aggregate allocations. Notice, however, that the representative agent's preferences $U(\cdot)$ differ from the individual consumer's preferences $u(\cdot)$ by construction.

1.4 Maliar and Maliar (RED 2003)

We now apply Constantinides's (1982) aggregation theorem by studying Maliar and Maliar (2001) and Maliar and Maliar (2003). They show that while Gorman aggregation fails, one can obtain a weaker aggregation result. The aggregate dynamics can be described by a representative agent, however the representative agent's preferences are different from preferences of the individual consumer. In particular, there exists a new preference shifter for the representative agent that depends on the distribution of individual productivity shocks. Therefore, the dynamics of aggregate variables depend on the distribution of shocks in Maliar and Maliar's (2003) economy.

Households: There is a unit continuum of infinitely lived agents, indexed by $i \in \mathcal{I} \equiv [0, 1]$. Let μ^i be the measure of type i agents such that $\int_{\mathcal{I}} d\mu_i = 1$.

Agents are endowed with a unit of time that can be used for labor or leisure. Preferences over consumption $\{c_t^i\}_{t=0}^{\infty}$ and labor $\{h_t^i\}_{t=0}^{\infty}$ paths are given by:

$$U = \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t u(c_t^i, 1 - h_t^i)$$

where the Bernoulli utility function is assumed to be separable and non-homothetic in labor:

$$u(c, 1 - h) = \frac{c^{1-\sigma}}{1 - \sigma} + \psi \frac{(1 - h)^{1-\gamma}}{1 - \gamma}$$

Agents receive idiosyncratic labor productivity shocks, $\varepsilon_t^i \in \mathcal{E}$, that are iid and have unit mean. Agents receive a wage w_t for each effective labor hour $\varepsilon_t^i h_t^i$ worked. Agents can insure against idiosyncratic shocks using Arrow Securities $a_{t+1}(\varepsilon)$, at price $p_t(\varepsilon)$, that pay one unit of consumption at $t + 1$ only if state ε is realized. Lastly, agents can save at interest rate r_t.

Household i's problem is:

$$\max_{\{c_t, h_t, k_{t+1}, a_{t+1}(\varepsilon)\}_{t=0}^{\infty}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left[\frac{(c_t^i)^{1-\sigma}}{1 - \sigma} + \psi \frac{(1 - h_t^i)^{1-\gamma}}{1 - \gamma} \right]$$

subject to

$$c_t^i + k_{t+1}^i + \int_{\mathcal{E}} p_t(\varepsilon) a_{t+1}^i(\varepsilon) d\varepsilon \leq w_t \varepsilon_t^i h_t^i + (1 + r_t) k_t^i + a_t^i(\varepsilon_t^i)$$

$$k_0^i, \ a_0^i \text{ given}$$

Firms: The representative firm possess a constant returns to scale production technology $F(K, L)$ that is strictly increasing and strictly concave in both K and L. Output is produced using $Y_t = z_t F(K_t, L_t)$ where z_t is an aggregate productivity shock. The representative firm rents capital at rate r_t and pays labor w_t, both prices taken as given. Capital depreciates at rate δ. Static profits are:

$$\Pi_t = z_t F(K_t, L_t) - w_t L_t - (r_t - \delta) K_t$$

Equilibrium: A competitive equilibrium is a household allocation $\{c_t^i, h_t, k_{t+1}^i, a_{t+1}^i(\varepsilon)\}_{t=0}^{\infty}$ for each i, a firm allocation $\{K_t, h_t, L_t\}_{t=0}^{\infty}$ and prices $\{w_t, h_t, r_t, p_t(\varepsilon)\}_{t=0}^{\infty}$ that (i) satisfy household optimality, (ii) satisfy firm optimality and (iii) satisfy market clearing conditions for capital, labor and resources:

$$K_t = \int_{\mathcal{I}} k_t^i d\mu^i$$

$$L_t = \int_{\mathcal{I}} \varepsilon_t^i h_t^i d\mu^i$$

$$\int_{\mathcal{I}} c_t^i d\mu^i + K_{t+1} = z_t F(K_t, L_t) + (1 - \delta) K_t$$

and Arrow securities are in net zero supply.

Planner's Allocations: Suppose we compute the Negishi Planner's allocation with Planner weights α^i. The static Negishi Planner's Problem given (C_t, L_t) is:

$$U(C_t, 1 - L_t) = \max_{\{c_t^i, h_t^i\}_{i \in \mathcal{I}}} \left\{ \int_{\mathcal{I}} \alpha^i u(c_t^i, h_t^i) d\mu^i \quad \text{s.t.} \quad \int_{\mathcal{I}} c_t^i d\mu^i \leq C_t, \ \int_{\mathcal{I}} \varepsilon_t^i h_t^i d\mu^i \leq L_t \right\}$$

Let the multipliers on the consumption and labor constraints be denoted by λ_t^c and λ_t^l, respectively. First order conditions with respect to a generic household i's consumption and labor give:

$$c_t^i = \left(\frac{\alpha^i}{\lambda_t^c} \right)^{\frac{1}{\sigma}}$$

$$h_t^i = 1 - \left(\frac{\psi \alpha^i}{\lambda_t^l \varepsilon_t^i} \right)^{\frac{1}{\gamma}}$$

Notice that consumption satisfies full risk sharing (as before) and that the planner makes more productive workers (higher ε_t^i) exert more labor hours.

Aggregating over agents, we obtain:

$$C_t = \int_{\mathcal{I}} c_t^i d\mu^i = (\lambda_t^c)^{-\frac{1}{\sigma}} \int_{\mathcal{I}} \left(\alpha^i\right)^{\frac{1}{\sigma}} d\mu^i$$

$$L_t = \int_{\mathcal{I}} \varepsilon_t^i h_t^i d\mu^i = 1 - \left(\frac{\psi}{\lambda_t^l}\right)^{\frac{1}{\gamma}} \int_{\mathcal{I}} \left(\alpha^i\right)^{\frac{1}{\gamma}} (\varepsilon_t^i)^{1-\frac{1}{\gamma}} d\mu^i$$

where we have used the assumption that ε has mean 1.

Then we can substitute aggregate consumption and labor into the policy functions to obtain:

$$c_t^i = \frac{\left(\alpha^i\right)^{\frac{1}{\sigma}}}{\int_{\mathcal{I}} \left(\alpha^i\right)^{\frac{1}{\sigma}} d\mu^i} C_t \tag{4}$$

$$1 - h_t^i = \frac{\left(\alpha^i\right)^{\frac{1}{\gamma}} \left(\varepsilon_t^i\right)^{-\frac{1}{\gamma}}}{\int_{\mathcal{I}} \left(\alpha^i\right)^{\frac{1}{\gamma}} (\varepsilon_t^i)^{1-\frac{1}{\gamma}} d\mu^i} (1 - L_t) \tag{5}$$

Representative Agent Construction: Now we appeal to Constantinides (1982). In the first component we write the Negishi Planner's Problem, substituting the policy functions $\{c_t^i, h_t^i\}$ from above:

$$\int_{\mathcal{I}} \alpha^i \left[\frac{(c_t^i)^{1-\sigma}}{1-\sigma} + \psi \frac{(1-h_t^i)^{1-\gamma}}{1-\gamma}\right] d\mu^i$$

$$= \int_{\mathcal{I}} \alpha^i \left[\left(\frac{\left(\alpha^i\right)^{\frac{1}{\sigma}}}{\int_{\mathcal{I}} \left(\alpha^i\right)^{\frac{1}{\sigma}} d\mu^i}\right)^{1-\sigma} \frac{(C_t)^{1-\sigma}}{1-\sigma} + \psi \left(\frac{\left(\alpha^i\right)^{\frac{1}{\gamma}} \left(\varepsilon_t^i\right)^{-\frac{1}{\gamma}}}{\int_{\mathcal{I}} \left(\alpha^i\right)^{\frac{1}{\gamma}} (\varepsilon_t^i)^{1-\frac{1}{\gamma}} d\mu^i}\right)^{1-\gamma} \frac{(1-L_t)^{1-\gamma}}{1-\gamma}\right] d\mu^i$$

$$= \frac{(C_t)^{1-\sigma}}{1-\sigma} + \psi \int_{\mathcal{I}} \alpha^i \left(\frac{\left(\alpha^i\right)^{\frac{1}{\gamma}} \left(\varepsilon_t^i\right)^{-\frac{1}{\gamma}}}{\int_{\mathcal{I}} (\alpha^i)^{\frac{1}{\gamma}} (\varepsilon_t^i)^{1-\frac{1}{\gamma}} d\mu^i}\right)^{1-\gamma} d\mu^i \frac{(1-L_t)^{1-\gamma}}{1-\gamma}$$

$$\equiv \frac{(C_t)^{1-\sigma}}{1-\sigma} + \Psi_t \frac{(1-L_t)^{1-\gamma}}{1-\gamma}$$

where we have defined

$$\Psi_t \equiv \psi \int_{\mathcal{I}} \frac{\left(\alpha^i\right)^{\frac{1}{\gamma}} \left(\varepsilon_t^i\right)^{1-\frac{1}{\gamma}}}{\left(\int_{\mathcal{I}} (\alpha^i)^{\frac{1}{\gamma}} (\varepsilon_t^i)^{1-\frac{1}{\gamma}} d\mu^i\right)^{1-\gamma}} d\mu^i = \psi \left(\int_{\mathcal{I}} \left(\alpha^i\right)^{\frac{1}{\gamma}} (\varepsilon_t^i)^{1-\frac{1}{\gamma}} d\mu^i\right)^{\gamma}$$

Hence, we have derived a analytical expression for the representative consumer's preferences. These preferences are nearly identical to the individual agent's preferences, except now the parameter that determines the level of utility from leisure depends on primitives of the individual's environment. In particular, Ψ_t depends on the distribution of idiosyncratic productivity shocks as well as the distribution of Negishi weights. If both distributions were degenerate (say, at 1) then the representative consumer would have the same preferences as the individual consumer. On the other hand if $\gamma > 1$, $\alpha^i = 1$ for all $i \in \mathcal{I}$, and the cross-sectional variance of ε_t^i increased, then by Jensen's inequality the representative agent's taste for leisure would increase.

Given the solution for the representative consumer's preferences, the resource constraint and initial capital K_0, we could solve for the representative agent's allocation of aggregate $\{C_t, L_t, K_{t+1}\}_{t=0}^{\infty}$ from:

$$\max_{\{C_t, L_t, K_{t+1}\}_{t=0}^{\infty}} \left\{ \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left[\frac{(C_t)^{1-\sigma}}{1-\sigma} + \Psi_t \frac{(1-L_t)^{1-\gamma}}{1-\gamma} \right] \quad \text{s.t.} \quad C_t + K_{t+1} = z_t F(K_t, L_t) + (1-\delta) K_t \right\}$$

1.5 Hsieh and Klenow (QJE 2009)

This is an exposition of both Hsieh and Klenow (2009) and Hsieh and Klenow (2014). The exposition demonstrates how to aggregate from the the firm-level when economy-wide output is a CES aggregate. We show how to construct a representative firm when individual firms are heterogeneous with respect to productivity and implicit taxes (wedges).

Final Goods: Let final output be given by a CES aggregator:

$$Y = \left(\sum_a \sum_i^{N_a} y_{ai}^{1-1/\sigma} \right)^{\frac{\sigma}{\sigma-1}}$$

where N_a is the number of firms of age a, and y_{ai} is a firm's output indexed by age and $i = 1, 2, \ldots, N_a$.

This implies a standard demand function (see Section 1.2):

$$y_{ai} = \left(\frac{p_{ai}}{P} \right)^{-\sigma} Y$$

where

$$P = \left(\sum_a \sum_i^{N_a} p_{ai}^{1-\sigma} \right)^{\frac{1}{1-\sigma}}$$

Production: Intermediate firms have a production function:

$$y_{ai} = z_{ai} k_{ai}^{\alpha} l_{ai}^{1-\alpha}$$

for each age a and firm $i = 1, \ldots, N_a$. Age and firm specific implicit taxes on output, labor payments and capital payments yield a profit function of the form:

$$\pi_{ai} = (1 - \tau_{ai}^{y}) p_{ai} y_{ai} - (1 + \tau_{ai}^{l}) w l_{ai} - (1 + \tau_{ai}^{k}) R k_{ai}$$

Social Planner Problem: We can apply the Constantinides (1982) approach to aggregation by defining two component problems. We will focus on the first, of choosing the allocation of factor inputs given aggregate quantities of capital and labor (K, L). In the second component, we could solve the dynamic problem of a representative firm with production technology $ZK^{\alpha}L^{1-\alpha}$, where Z is an endogenous outcome. We will also characterize Z.

Cost Minimization: Given taxes and an output demand y, a type (a, i) firm solves:

$$C(y) = \min_{k,l} (1 + \tau^l) w l + (1 + \tau^k) R k$$

$$\text{s.t.} \quad z k^{\alpha} l^{1-\alpha} \geq y$$

First order conditions give:

$$\frac{(1 + \tau^l) w l}{1 - \alpha} = \frac{(1 + \tau^k) R k}{\alpha}$$

and the constraint gives:

$$k = \left(\frac{y}{z} \right)^{\frac{1}{\alpha}} \left(\frac{1}{l} \right)^{\frac{1-\alpha}{\alpha}}$$

$$l = \left(\frac{y}{z} \right)^{\frac{1}{1-\alpha}} \left(\frac{1}{k} \right)^{\frac{\alpha}{1-\alpha}}$$

Substituting the constraint into the FOC gives:

$$(1+\tau^k)Rk = m \cdot \alpha y$$

$$(1+\tau^l)wl = m \cdot (1-\alpha)y$$

where we define marginal cost as:

$$\boxed{m \triangleq \left(\frac{1}{z}\right)\left(\frac{R}{\alpha}\right)^{\alpha}\left(\frac{w}{1-\alpha}\right)^{1-\alpha}\left(1+\tau^k\right)^{\alpha}\left(1+\tau^l\right)^{1-\alpha}}$$

It will be convenient to define common (ψ) and idiosyncratic (ω) terms in the marginal cost:

$$\left(\frac{m}{1-\tau^y}\right) \triangleq \psi \cdot \left(\frac{\omega}{z}\right)$$

Therefore the cost function for the firm is:

$$C(y) = m \cdot y$$

Profit Maximization: Given the cost minimizing factor allocation and the consumer's demand function, the firm chooses output to maximize profits. The demand function implies that firm revenues are:

$$p(y)y = PY^{1/\sigma}y^{1-1/\sigma}$$

and therefore profits can be written:

$$\pi = \max_y \; (1-\tau^y)PY^{1/\sigma}y^{1-1/\sigma} - m \cdot y$$

First order conditions give:

$$m \cdot y = (1-1/\sigma)(1-\tau^y)PY^{1/\sigma}y^{1-1/\sigma}$$

Let $\mu \triangleq \sigma/(\sigma-1)$ denote the markup. First order conditions imply:

$$y = m^{-\sigma}\left(\frac{1}{\mu}(1-\tau^y)PY^{1/\sigma}\right)^{\sigma}$$

$$= (PY^{1/\sigma})^{\sigma} \cdot \mu^{-\sigma}\left(\frac{m}{1-\tau^y}\right)^{-\sigma}$$

$$
\begin{aligned}
&= (PY^{1/\sigma})^\sigma \cdot \left(\mu\psi \cdot \frac{\omega}{z}\right)^{-\sigma} \\
p &= PY^{1/\sigma} y^{-1/\sigma} \\
&= (PY^{1/\sigma}) \left[\left(PY^{1/\sigma}\right)^\sigma \left(\mu \cdot \frac{m}{1-\tau^y}\right)^{-\sigma} \right]^{-1/\sigma} \\
&= \mu \cdot \left(\frac{m}{1-\tau^y}\right) \\
&= \mu\psi \cdot \frac{\omega}{z}
\end{aligned}
$$

Therefore we can put this information together as:

$$
\begin{aligned}
m &= (1-\tau^y) \cdot \left(\frac{\psi\omega}{z}\right) \\
p &= \mu \cdot \left(\frac{\psi\omega}{z}\right) \\
y &= (PY^{1/\sigma})^\sigma \cdot \left(\mu \cdot \frac{\psi\omega}{z}\right)^{-\sigma} \\
py &= (PY^{1/\sigma})^\sigma \cdot \left(\mu \cdot \frac{\psi\omega}{z}\right)^{1-\sigma} \\
\pi &= (1/\sigma)(1-\tau^y) py
\end{aligned}
$$

Firm's Allocation: Given the firm's output decision, we can now fully characterize the firm's factor allocation. The cost function is:

$$
\begin{aligned}
C(y) = m \cdot y &= (1-\tau^y)\frac{\psi\omega}{z} \cdot (PY^{1/\sigma})^\sigma \left(\mu \cdot \frac{\psi\omega}{z}\right)^{-\sigma} \\
&= (1-\tau^y)\mu^{-\sigma}(PY^{1/\sigma})^\sigma \left(\frac{\psi\omega}{z}\right)^{1-\sigma}
\end{aligned}
$$

and therefore the factor payments are:

$$
(1+\tau^k)Rk = \alpha(1-\tau^y)\left(\frac{PY^{1/\sigma}}{\mu}\right)^\sigma \left(\frac{\psi\omega}{z}\right)^{1-\sigma}
$$

$$(1+\tau^l)wl = (1-\alpha)(1-\tau^y)\left(\frac{PY^{1/\sigma}}{\mu}\right)^{\sigma}\left(\frac{\psi\omega}{z}\right)^{1-\sigma}$$

Rewriting the above optimality condition, we obtain:

$$k = \left(\frac{PY^{1/\sigma}}{\mu}\right)^{\sigma}\psi^{1-\sigma}\left(\frac{\alpha}{R}\right)\cdot\left(\frac{1-\tau^y}{1+\tau^k}\right)\left(\frac{\omega}{z}\right)^{1-\sigma}$$

$$l = \left(\frac{PY^{1/\sigma}}{\mu}\right)^{\sigma}\psi^{1-\sigma}\left(\frac{1-\alpha}{w}\right)\cdot\left(\frac{1-\tau^y}{1+\tau^l}\right)\left(\frac{\omega}{z}\right)^{1-\sigma}$$

Then let's check that the firm's output is as given above:

$$\begin{aligned}
y &= z\left(\frac{PY^{1/\sigma}}{\mu}\right)^{\sigma}\psi^{1-\sigma}\left(\frac{\alpha}{R}\cdot\frac{1-\tau^y}{1+\tau^k}\right)^{a}\left(\frac{1-\alpha}{w}\cdot\frac{1-\tau^y}{1+\tau^l}\right)^{1-\alpha}\left(\frac{\omega}{z}\right)^{1-\sigma} \\
&= z\left(\frac{PY^{1/\sigma}}{\mu}\right)^{\sigma}\psi^{1-\sigma}\cdot\Big(\psi\omega\Big)^{-1}\cdot\left(\frac{\omega}{z}\right)^{1-\sigma} \\
&\overset{\checkmark}{=}\left(\frac{PY^{1/\sigma}}{\mu\psi}\right)^{\sigma}\cdot\left(\frac{\omega}{z}\right)^{-\sigma}
\end{aligned}$$

where, recall, we defined:

$$\omega \triangleq \frac{(1+\tau^k)^{\alpha}(1+\tau^l)^{1-\alpha}}{(1-\tau^y)}$$

Aggregation: Using these expressions for the firms' decision rules, let's find aggregate variables. To start, find the price index as:

$$Y = \left(\sum_a\sum_i\left[\left(\frac{PY^{1/\sigma}}{\mu\psi}\right)^{\sigma}\cdot\left(\frac{\omega_{ai}}{z_{ai}}\right)^{-\sigma}\right]^{1-1/\sigma}\right)^{\frac{\sigma}{\sigma-1}}$$

$$Y = \left(\frac{PY^{1/\sigma}}{\mu\psi}\right)^{\sigma}\left(\sum_a\sum_i\left(\frac{\omega_{ai}}{z_{ai}}\right)^{1-\sigma}\right)^{\frac{\sigma}{\sigma-1}}$$

$$P = \mu\psi\left(\sum_a\sum_i\left(\frac{\omega_{ai}}{z_{ai}}\right)^{1-\sigma}\right)^{\frac{1}{1-\sigma}}$$

$$P \overset{\checkmark}{=}\left(\sum_a\sum_i p_{ai}^{1-\sigma}\right)^{\frac{1}{1-\sigma}}$$

Then aggregate revenues are:

$$\sum_a \sum_i p_{ai} y_{ai} = \left(PY^{1/\sigma}\right)^{\sigma} (\mu\psi)^{1-\sigma} \sum_a \sum_i \left(\frac{\omega_{ai}}{z_{ai}}\right)^{1-\sigma}$$
$$= \left(PY^{1/\sigma}\right)^{\sigma} P^{1-\sigma}$$
$$= PY$$

Aggregate capital and labor is simply:

$$K = \left(PY^{1/\sigma}\right)^{\sigma} (\mu\psi)^{1-\sigma} \left(\frac{\alpha}{\mu R}\right) \cdot \left[\sum_a \sum_i \left(\frac{1-\tau_{ai}^{y}}{1+\tau_{ai}^{k}}\right) \left(\frac{\omega_{ai}}{z_{ai}}\right)^{1-\sigma}\right]$$
$$L = \left(PY^{1/\sigma}\right)^{\sigma} (\mu\psi)^{1-\sigma} \left(\frac{1-\alpha}{\mu w}\right) \cdot \left[\sum_a \sum_i \left(\frac{1-\tau_{ai}^{y}}{1+\tau_{ai}^{l}}\right) \left(\frac{\omega_{ai}}{z_{ai}}\right)^{1-\sigma}\right]$$

Allocation Across Firms: Given the firms' decision rules and aggregate variables, we can derive each firm's factor and output choice as a share of total factors and total output.

Firms' shares of aggregate revenues are:

$$\frac{p_{ai} y_{ai}}{PY} = \frac{\left(PY^{1/\sigma}\right)^{\sigma} (\mu\psi)^{1-\sigma} \left(\frac{\omega_{ai}}{z_{ai}}\right)^{1-\sigma}}{\mu\psi \left(\sum_a \sum_i \left(\frac{\omega_{ai}}{z_{ai}}\right)^{1-\sigma}\right)^{\frac{1}{1-\sigma}} \cdot \left(\frac{PY^{1/\sigma}}{\mu\psi}\right)^{\sigma} \left(\sum_a \sum_i \left(\frac{\omega_{ai}}{z_{ai}}\right)^{1-\sigma}\right)^{\frac{\sigma}{\sigma-1}}}$$
$$= \frac{\left(z_{ai}/\omega_{ai}\right)^{\sigma-1}}{\sum_a \sum_i \left(z_{ai}/\omega_{ai}\right)^{\sigma-1}}$$

Firms' output relative to aggregate output and price relative to the price index are:

$$\frac{y_{ai}}{Y} = \left(\frac{\left(\omega_{ai}/z_{ai}\right)}{\left(\sum_a \sum_i \left(\omega_{ai}/z_{ai}\right)^{1-\sigma}\right)^{\frac{1}{1-\sigma}}}\right)^{-\sigma}$$
$$\frac{p_{ai}}{P} = \frac{\left(\omega_{ai}/z_{ai}\right)}{\left(\sum_a \sum_i \left(\omega_{ai}/z_{ai}\right)^{1-\sigma}\right)^{\frac{1}{1-\sigma}}}$$

Capital and labor shares are:

$$k_{ai} = \frac{\left(\frac{1-\tau^y_{ai}}{1+\tau^k_{ai}}\right)\left(\frac{\omega_{ai}}{z_{ai}}\right)^{1-\sigma}}{\sum_a \sum_i \left(\frac{1-\tau^y_{ai}}{1+\tau^k_{ai}}\right)\left(\frac{\omega_{ai}}{z_{ai}}\right)^{1-\sigma}} \cdot K$$

$$l_{ai} = \frac{\left(\frac{1-\tau^y_{ai}}{1+\tau^l_{ai}}\right)\left(\frac{\omega_{ai}}{z_{ai}}\right)^{1-\sigma}}{\sum_a \sum_i \left(\frac{1-\tau^y_{ai}}{1+\tau^l_{ai}}\right)\left(\frac{\omega_{ai}}{z_{ai}}\right)^{1-\sigma}} \cdot L$$

Using the definition of py/PY we can rewrite the capital and labor shares in terms of revenue shares:

$$k_{ai} = \frac{\left(\frac{1-\tau^y_{ai}}{1+\tau^k_{ai}}\right)\left(\frac{p_{ai}y_{ai}}{PY}\right)}{\sum_a \sum_i \left(\frac{1-\tau^y_{ai}}{1+\tau^k_{ai}}\right)\left(\frac{p_{ai}y_{ai}}{PY}\right)} \cdot K$$

$$l_{ai} = \frac{\left(\frac{1-\tau^y_{ai}}{1+\tau^l_{ai}}\right)\left(\frac{p_{ai}y_{ai}}{PY}\right)}{\sum_a \sum_i \left(\frac{1-\tau^y_{ai}}{1+\tau^l_{ai}}\right)\left(\frac{p_{ai}y_{ai}}{PY}\right)} \cdot L$$

Aggregate Productivity: Aggregate productivity can be derived from the above capital and labor shares. Suppose that aggregate output can be expressed as $Y = ZK^\alpha L^{1-\alpha}$, where aggregate productivity is Z. Since an individual firm's output is given by $y_{ai} = z_{ai} k^\alpha_{ai} l^{1-\alpha}_{ai}$ we can write:

$$y_{ai} = z_{ai} \cdot \frac{\left(\frac{1-\tau^y_{ai}}{1+\tau^k_{ai}}\right)^\alpha \left(\frac{1-\tau^y_{ai}}{1+\tau^l_{ai}}\right)^{1-\alpha} \left(\frac{\omega_{ai}}{z_{ai}}\right)^{1-\sigma}}{\left(\sum_a \sum_i \left(\frac{1-\tau^y_{ai}}{1+\tau^k_{ai}}\right)\left(\frac{\omega_{ai}}{z_{ai}}\right)^{1-\sigma}\right)^\alpha \left(\sum_a \sum_i \left(\frac{1-\tau^y_{ai}}{1+\tau^l_{ai}}\right)\left(\frac{\omega_{ai}}{z_{ai}}\right)^{1-\sigma}\right)^{1-\alpha}} \cdot K^\alpha L^{1-\alpha}$$

$$y_{ai} \triangleq \hat{\Omega} \left(\omega_{ai}/z_{ai}\right)^{-\sigma} \cdot K^\alpha L^{1-\alpha}$$

$$Y = \hat{\Omega} \left(\sum_a \sum_i \left(\frac{z_{ai}}{\omega_{ai}}\right)^{\sigma-1}\right)^{\frac{\sigma}{\sigma-1}} \cdot K^\alpha L^{1-\alpha}$$

In order to write TFP in the same way as Hsieh and Klenow do, substitute the revenue share into the denominator:

$$\begin{aligned}
\hat{\Omega}^{-1} &= \left(\sum_a \sum_i \left(\frac{1-\tau_{ai}^y}{1+\tau_{ai}^k}\right)\left(\frac{\omega_{ai}}{z_{ai}}\right)^{1-\sigma}\right)^{\alpha}\left(\sum_a \sum_i \left(\frac{1-\tau_{ai}^y}{1+\tau_{ai}^l}\right)\left(\frac{\omega_{ai}}{z_{ai}}\right)^{1-\sigma}\right)^{1-\alpha} \\
&= \left(\sum_a \sum_i \left(\frac{1-\tau_{ai}^y}{1+\tau_{ai}^k}\right)\left(\frac{p_{ai}y_{ai}}{PY}\right)\right)^{\alpha}\left(\sum_a \sum_i \left(\frac{1-\tau_{ai}^y}{1+\tau_{ai}^l}\right)\left(\frac{p_{ai}y_{ai}}{PY}\right)\right)^{1-\alpha} \cdot \left(\sum_a \sum_i \left(\frac{\omega_{ai}}{z_{ai}}\right)^{1-\sigma}\right) \\
&\triangleq \Omega^{-1} \cdot \sum_a \sum_i \left(\frac{\omega_{ai}}{z_{ai}}\right)^{1-\sigma}
\end{aligned}$$

Therefore:

$$Y = \Omega \left(\sum_a \sum_i \left(\frac{z_{ai}}{\omega_{ai}}\right)^{\sigma-1}\right)^{\frac{1}{\sigma-1}} \cdot K^{\alpha} L^{1-\alpha}$$

$$Z \triangleq \left(\sum_a \sum_i \left(z_{ai} \cdot \frac{\Omega}{\omega_{ai}}\right)^{\sigma-1}\right)^{\frac{1}{\sigma-1}}$$

$$\Omega \triangleq \left[\left(\sum_a \sum_i \left(\frac{1-\tau_{ai}^y}{1+\tau_{ai}^k}\right)\left(\frac{p_{ai}y_{ai}}{PY}\right)\right)^{\alpha}\left(\sum_a \sum_i \left(\frac{1-\tau_{ai}^y}{1+\tau_{ai}^l}\right)\left(\frac{p_{ai}y_{ai}}{PY}\right)\right)^{1-\alpha}\right]^{-1}$$

Distortions: Resource allocation is driven by distortions rather than differences in TFP. If TFP is given by the standard Solow residual:

$$TFPQ_{ai} \triangleq \frac{y_{ai}}{k_{ai}^{\alpha} l_{ai}^{1-\alpha}} = z_{ai}$$

then the revenue based TFP is given by:

$$TFPR_{ai} \triangleq \frac{p_{ai}y_{ai}}{k_{ai}^{\alpha} l_{ai}^{1-\alpha}} = p_{ai}z_{ai}$$

measures the effect of distortions on measured TFP. The revenue based measure will reflect differences in the marginal revenue products of labor and capital across firms:

$$MRPL_{ai} \triangleq (1-\alpha)\mu^{-1} \cdot \frac{p_{ai}y_{ai}}{l_{ai}} = w \cdot \frac{1+\tau_{ai}^l}{1-\tau_{ai}^y}$$

$$MRPK_{ai} \triangleq \alpha\mu^{-1} \cdot \frac{p_{ai}y_{ai}}{k_{ai}} = R \cdot \frac{1+\tau_{ai}^k}{1-\tau_{ai}^y}$$

These definitions imply that:

$$\begin{aligned} TFPR_{ai} &= \frac{p_{ai}y_{ai}}{k_{ai}^{\alpha}l_{ai}^{1-\alpha}} \\ &= \left(\frac{\mu}{\alpha}\cdot MRPK_{ai}\right)^{\alpha}\left(\frac{\mu}{1-\alpha}\cdot MRPL_{ai}\right)^{1-\alpha} \\ &= \left(\frac{\mu}{\alpha}\cdot R\cdot\frac{1+\tau_{ai}^{k}}{1-\tau_{ai}^{y}}\right)^{\alpha}\left(\frac{\mu}{1-\alpha}\cdot w\cdot\frac{1+\tau_{ai}^{l}}{1-\tau_{ai}^{y}}\right)^{1-\alpha} \\ &= \mu\psi\omega_{ai} \end{aligned}$$

Discussion: We have used Hsieh and Klenow's (2009) setup to demonstrate how to aggregate (using a homothetic technology) from the firm-level. We showed how distortions at the firm level affect the aggregate representation of the economy. In particular, distortions manifest as the Solow residual of the aggregate technology.

2 Permanent Income Hypothesis

In this section, we will review the empirical evidence on asset market completeness and appeal to exogenous restrictions on asset trade to reconcile theory with evidence. We will then consider the "strict" Permanent Income Hypothesis model to understand the data. First we develop the theoretical implications of the model and then we review the literature that has empirically tested PIH's predictions.

2.1 Asset Markets Restrictions

In the last section we worked with complete market models and derived a set of aggregation results. We now examine the empirical applicability of the complete market benchmark.

Preliminaries: We develop some notation for discussing uncertainty. Let s_t denote the state of the economy at time t. Let the set of possible states be S_t such that $s_t \in S_t$. Denote the history of states up to time t as $s^t - \{s_0, \ldots, s_t\} \in S^t \equiv \times_{\tau=0}^{t} S_\tau$. The particular history s^t occurs with probability $\pi(s^t)$. Lastly, denote individual i's income following history s^t by $y_t^i(s^t)$.

Autarky: In the absence of asset markets (or even a storage technology for intertemporal substitution of resources), consumption is afforded directly from income:

$$c_t^i(s^t) = y_t^i(s^t)$$

Therefore, agents do not share or insulate against income risk under autarky, for they do not have access to any market or instrument by which they could do so.

Complete Markets: Now suppose that agents have access to Arrow securities, assets that pay a unit of consumption if state s_{t+1} occurs and zero otherwise. The price of an Arrow security is denoted $q(s_{t+1}, s^t)$, while the quantity of is denoted $a_{t+1}^i(s_{t+1}, s^t)$.

It can be easily verified that a complete markets economy with time-0 Arrow-Debreu prices in Section 1.3

$$\sum_{t=0}^{\infty} \sum_{s^t \in S^t} p_t(s^t) c_t^i(s^t) \leq \sum_{t=0}^{\infty} \sum_{s^t \in S^t} p_t(s^t) y_t^i(s^t)$$

delivers the same allocation as a complete markets economy in which agents face a sequential budget constraint and trade Arrow securities,

$$c_t^i(s^t) + \sum_{s^t \in S^t} q_t(s_{t+1}, s^t) a_{t+1}^i(s_{t+1}, s^t) \leq y_t^i(s^t) + a_t^i(s^t)$$

as long as a no Ponzi condition is satisfied:

$$\lim_{t \to \infty} \sum_{s^t \in S^t} q_t(s_{t+1}, s^t) a_{t+1}^i(s_{t+1}, s^t) \geq 0$$

See chapter 8 of Ljungqvist and Sargent (2012) for a proofs of equivalence. Given the equivalence between the two problems, we know that full risk sharing results, c.f. equations (2) and (3). That is, given Planner weights $\{\alpha^i\}_{i \in \mathcal{I}}$,

$$\frac{u'(c_t^i(s^t))}{u'(c_t^j(s^t))} = \frac{\alpha^j}{\alpha^i}$$

and the ratio of marginal utility across each set of agents is constant for each history of states.

Notice that if the utility function takes an isoelastic form, such as $u(c) = c^{1-\sigma}/(1-\sigma)$, then individual consumption is a constant share of aggregate consumption (as in equation (4) above):

$$c_t^i(s^t) = \frac{(\alpha^i)^{\frac{1}{\sigma}}}{\sum_{j \in \mathcal{I}} (\alpha^j)^{\frac{1}{\sigma}}} \, C_t(s^t)$$

where $C_t(s^t) = \sum_{i \in \mathcal{I}} c_t^i(s^t)$ is aggregate consumption. Because individual agents may not have perfectly orthogonal income processes, there may be some amount of uninsurable aggregate uncertainty in income across agents. As a result, consumption is not constant and aggregate consumption can fluctuate, even in the presence of complete markets. Complete markets only pins down individual shares of aggregate consumption.

Empirical Evaluation: Taking log differences, we can check how correlated individual consumption growth is with aggregate consumption growth:

$$\log\left(c_t^i / c_{t-1}^i\right) = \beta \log\left(C_t / C_{t-1}\right) + \varepsilon_t^i$$

We could also follow Mace (1991) and check the relationship with individual income as well:

$$\log\left(c_t^i / c_{t-1}^i\right) = \beta_1 \log\left(C_t / C_{t-1}\right) + \beta_2 \log\left(y_t^i / y_{t-1}^i\right) + \varepsilon_t^i$$

If $\beta_1 = 1$ while $\beta_2 = 0$ then the data support full risk sharing. On the other hand, if $\beta_1 = 0$ and $\beta_2 = 1$ then the data implies autarkic outcomes. The consensus is that the data support some intermediate or partial risk sharing.

Asset Market Restrictions: There are alternative asset market arrangements that support partial risk sharing in equilibrium. These market arrangements are commonly referred to as *incomplete markets*. We will study *endogenously incomplete markets* later in the course (in the section on "Consumer Finance and Insurance").

For now we will consider *exogenously incomplete markets*, which consist of an exogenous restriction on the set of state-contingent contracts that agents may trade. In particular, we restrict securities to be non-contingent. This means that the Arrow security $a_{t+1}^i(s_{t+1}, s^t)$ that could be traded for each $s_{t+1} \in S_{t+1}$ at price $q_t(s_{t+1}, s^t)$ is now exogenously restricted to a single asset that can be traded against all states simultaneously, $a_{t+1}^i(s^t)$ at price $q_t(s^t)$. Therefore the sequential budget constraint becomes:

$$c_t^i(s^t) + q_t(s^t) a_{t+1}^i(s^t) \leq y_t^i(s^t) + a_t^i(s^t)$$

which can be written more parsimoniously by suppressing state-dependence (which we apply in the following sections).

2.2 Canonical Consumption Savings Problem

In what follows, we will consider a consumption-savings model that takes the following canonical form:

$$
\begin{aligned}
v^i(a_0^i, y_0^i) = \max_{\{c_t^i, a_{t+1}^i\}_{t=0}^{\infty}} \ & \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t u(c_t^i) \\
\text{s.t.} \quad c_t^i + \frac{1}{1+r_t} a_{t+1}^i &\leq y_t^i + a_t^i \\
a_{t+1}^i &\geq \underline{a}_{t+1}^i \\
c_t^i &\geq 0
\end{aligned}
$$

where $q_t \equiv 1/(1+r_t)$ and r_t is the interest rate on the risk-free bond at time t, y_t^i follows some stochastic process, $\underline{a}_{t+1}^i$ is agent i's borrowing limit at time t for funds to be repaid at $t+1$, and $v^i(a_0^i, y_0^i)$ is the net present value of an agent that is endowed with initial assets a_0^i and initial income y_0^i.

We can rewrite consumption-savings problem in recursive form:

$$
\begin{aligned}
v(a_t, y_t) = \max_{c_t, a_{t+1}} \ & u(c_t) + \beta \mathbb{E}_t \left[v(a_{t+1}, y_{t+1}) \right] \\
\text{s.t.} \quad c_t + \frac{1}{1+r_t} a_{t+1} &\leq y_t + a_t \\
a_{t+1} &\geq \underline{a}_{t+1} \\
c_t &\geq 0
\end{aligned}
$$

where we have suppressed the dependence of variables on i, since it is clear that we are considering an particular agent's problem. What is necessary to understand is that the individual agent has a state vector (a_t, y_t).

2.3 "Strict" Permanent Income Hypothesis

We will now specialize the canonical consumption-savings problem along the lines of Deaton (1992), by making particular assumptions on preferences, technologies and parameters. The specialization we consider is a strict version of the Permanent Income Hypothesis (PIH), which was first formalized by Milton Friedman and Franco Modigliani. The "Strict" PIH model features:

- Quadratic utility specification:

$$u(c) = -\frac{\alpha}{2}(c_t - \bar{c})^2$$

  where $\bar{c}$ is a "bliss point" of maximium utility and α is a utility parameter.

- One-period bond returns are certain and pinned down by the discount rate such that:

$$\beta(1 + r) = 1$$

- Borrowing constraints are replaced by the No Ponzi Condition for all $t \geq 0$:

$$\mathbb{E}_t \left[\lim_{j \to \infty} \left(\frac{1}{1 + r} \right)^j a_{t+j} \right] \geq 0 \tag{6}$$

We will now characterize the theoretical properties of the "strict" PIH model.

Optimal Consumption: Taking first order conditions, we obtain the Euler equation:

$$u'(c) \geq \beta(1 + r)\mathbb{E}_t \left[u'(c_{t+1}) \right]$$

which, given quadratic utility, $\beta(1 + r) = 1$ and the No Ponzi condition, is now specialized to:

$$\alpha\bar{c} - \alpha c_t = \mathbb{E}_t[\alpha\bar{c} - \alpha c_{t+1}] \qquad \Rightarrow \qquad c_t = \mathbb{E}_t[c_{t+1}]$$

This is Hall's (1978) result that consumption is a *random walk* or *martingale*, and by the Law of Iterated Expectations:

$$c_t = \mathbb{E}_t[c_{t+j}] \quad \forall\ j \geq 0$$

Permanent Income: Following Deaton (1992) define human wealth at time t as:

$$h_t \equiv \sum_{j=0}^{\infty} \left(\frac{1}{1 + r} \right)^j \mathbb{E}_t[y_{t+j}]$$

which is discounted expected income. Define financial wealth as a_t and total wealth as $a_t + h_t$. Then define *permanent income* as the annuity value of total wealth: $\frac{r}{1+r}(a_t + h_t)$.

Now we will prove that *consumption equals permanent income.* To do so, iterate forward on the budget constraint and divide by $(1+r)^j$ for each iteration j:

$$
\begin{aligned}
c_t &= y_t + a_t - \left(\frac{1}{1+r}\right) a_{t+1} \\
\left(\frac{1}{1+r}\right) c_{t+1} &= \left(\frac{1}{1+r}\right) y_{t+1} + \left(\frac{1}{1+r}\right) a_{t+1} - \left(\frac{1}{1+r}\right)^2 a_{t+2} \\
&\vdots \qquad\qquad \vdots \qquad\qquad \vdots \\
\left(\frac{1}{1+r}\right)^j c_t &= \left(\frac{1}{1+r}\right)^j y_{t+j} + \left(\frac{1}{1+r}\right)^j a_{t+j} - \left(\frac{1}{1+r}\right)^{j+1} a_{t+1+j}
\end{aligned}
$$

Summing gives:

$$
\sum_{j=0}^{\infty} \left(\frac{1}{1+r}\right)^j c_{t+j} = a_t + \sum_{j=0}^{\infty} \left(\frac{1}{1+r}\right)^j y_{t+j} - \lim_{j\to\infty} \left(\frac{1}{1+r}\right)^j a_{t+j}
$$

Taking expectations and applying the No Ponzi Condition yields:

$$
\sum_{j=0}^{\infty} \left(\frac{1}{1+r}\right)^j \mathbb{E}_t[c_{t+j}] = a_t + \sum_{j=0}^{\infty} \left(\frac{1}{1+r}\right)^j \mathbb{E}_t[y_{t+j}]
$$

But because consumption is a random walk, $\mathbb{E}_t[c_{t+j}] = c_t$ for all $j \geq 0$:

$$
\begin{aligned}
c_t &= \frac{r}{1+r}\left(a_t + \sum_{j=0}^{\infty}\left(\frac{1}{1+r}\right)^j \mathbb{E}_t[y_{t+j}]\right) \qquad (7) \\
&\equiv \frac{r}{1+r}\left(a_t + h_t\right)
\end{aligned}
$$

Therefore, consumption equals permanent income, as we sought to show.

Certainty Equivalence: Notice that c_t only depends on the expected value of income, but does not depend on the variance, skewness, kurtosis, etc. Therefore, any stochastic processes for income that deliver the same expected value are equivalent to the consumer from the perspective of consumption decisions. In particular, consider the alternative income process $\{\hat{y}_t\}_{t=0}^{\infty}$ that gives $\hat{y}_t = h_t \equiv \sum_{j\geq 0}(1+r)^{-j}\mathbb{E}_t[y_{t+j}]$ with certainty. Since $\hat{h}_t = h_t$ by construction, the agent is indifferent between a risky income process and the certain process $\hat{y}$. This result is due to the quadratic utility specification, which implies linear marginal utility of consumption.

Consumption and Wealth Dynamics: First, let's consider consumption dynamics. Using the martingale property and the Law of Iterated Expectations, we can write:

$$\begin{aligned}
\Delta c_t &= c_t - c_{t-1} \\
&= c_t - \mathbb{E}_{t-1} c_t \\
&= \frac{r}{1+r}\Big((a_t + h_t) - \mathbb{E}_{t-1}\left[(a_t + h_t)\right]\Big) \\
&= \frac{r}{1+r}\left(a_t - \mathbb{E}_{t-1}[a_t] + \sum_{j=0}^{\infty}\left(\frac{1}{1+r}\right)^j \Big(\mathbb{E}_t[y_{t+j}] - \mathbb{E}_{t-1}[\mathbb{E}_t[y_{t+j}]]\Big)\right) \\
\Delta c_t &= \frac{r}{1+r}\sum_{j=0}^{\infty}\left(\frac{1}{1+r}\right)^j \Big(\mathbb{E}_t[y_{t+j}] - \mathbb{E}_{t-1}[y_{t+j}]\Big) \qquad (8)
\end{aligned}$$

Therefore, the change in consumption between times $t-1$ and t is proportional to new information agents receive about their discounted expected income.

Turning to wealth dynamics, we wish to compute Δa_{t+1}. To do so, we will substitute the budget constraint and optimal consumption choice:

$$\begin{aligned}
\Delta a_{t+1} &= a_{t+1} - a_t \\
&= r a_t + (1+r)(y_t - c_t) \\
&= r a_t + (1+r) y_t - r(a_t + h_t) \\
&= (1+r) y_t - r y_t - r\sum_{j=1}^{\infty}\left(\frac{1}{1+r}\right)^j \mathbb{E}_t[y_{t+j}] \\
&= y_t - \sum_{j=1}^{\infty}\left(\frac{1}{1+r}\right)^j \Big((1+r) - 1\Big)\mathbb{E}_t[y_{t+j}] \\
&= y_t - \sum_{j=1}^{\infty}\left(\frac{1}{1+r}\right)^{j-1} \mathbb{E}_t[y_{t+j}] + \sum_{j=1}^{\infty}\left(\frac{1}{1+r}\right)^j \mathbb{E}_t[y_{t+j}] \\
&= -\left(\sum_{j=1}^{\infty}\left(\frac{1}{1+r}\right)^{j-1} \mathbb{E}_t[y_{t+j}] - \sum_{j=0}^{\infty}\left(\frac{1}{1+r}\right)^j \mathbb{E}_t[y_{t+j}]\right)
\end{aligned}$$

$$= -\sum_{j=1}^{\infty}\left(\frac{1}{1+r}\right)^{j-1}(\mathbb{E}_t[y_{t+j}] - \mathbb{E}_t[y_{t-1+j}])$$

$$\frac{1}{1+r}\Delta a_{t+1} = -\sum_{j=1}^{\infty}\left(\frac{1}{1+r}\right)^{j}\mathbb{E}_t[\Delta y_{t+j}] \tag{9}$$

This shows that the change in savings is proportional to the inverse of discounted expected income change. In other words, agents use savings to offset expected income fluctuations. If income is anticipated to rise, say deterministically by $\Delta y_{t+j} = (1+g)^{j-1}$ for $g < r$, then savings should decrease by:

$$\Delta a_{t+1} = -\sum_{j=1}^{\infty}\left(\frac{1+g}{1+r}\right)^{j-1} = -\frac{1+r}{r-g}$$

where an increase in income growth increases the rate at which the consumer decreases savings.

2.4 Empirically Evaluating the PIH

This section reviews work that empirically evaluates the Permanent Income Hypothesis.[2] We focus on the so-called *excess sensitivity* and *excess smoothness* puzzles as well attempts to reconcile the theory with these puzzles for the PIH model.

Excess Sensitivity: Hall (1978) first showed that if consumers are forward looking, have rational expectations, evaluate consumption streams utilizing a quadratic utility function, and can freely borrow and lend at a constant interest rate, then consumption follows a random walk. Hall (1978) tested the whether consumption follows a random walk with a regression of the form:

$$c_t = \gamma_0 + \gamma_1 c_{t-1} + \gamma_2 z_{t-1}$$

where z_{t-1} can be any variable known at time $t-1$. If consumption follows a random walk, then the regression should identify $\gamma_1 = 1$ and $\gamma_2 = 0$. While Hall's (1978) results show $\gamma_1 \approx 1$, he also found an estimate for γ_2 that was significantly larger than zero when z_{t-1} was measured as aggregate stock market returns.

Flavin (1981) tested the related prediction that if consumption is a random walk, then it

[2] This section borrows extremely heavily from Chris Carroll's lecture notes. I recommend you read his lecture notes to get a fuller picture of this topic. Another useful reference is Deaton (1992).

can be represented as:

$$c_{t+1} = c_t + \varepsilon_{t+1}$$

with $\mathbb{E}_t[\varepsilon_{t+1}] = 0$ and $cov(c_t, \varepsilon_{t+1}) = 0$. Furthermore, theory predicts $cov(\varepsilon_{t+1}, z_t) = 0$ for any variable known at the previous period. Flavin (1981) estimated the relation:

$$\Delta c_t = \mu_0 + \mu_1 z_{t-1}$$

Placed in the context of Hall's (1978) work, Flavin (1981) imposed the restriction $\gamma_1 = 1$ and found μ_1 significantly different from zero when current growth in consumption is regressed on lagged aggregate income growth or stock market returns. For example, Deaton (1992) reports on page 94:

$$\begin{array}{cccccc} \Delta c_t & = & 11.39 & + & 0.121 & \Delta y_{t-1} \\ & & (9.7) & & (3.20) & \end{array} \tag{10}$$

Therefore, consumption growth is predicable by lagged income growth, which confronts the PIH theory with a puzzle since PIH predicts consumption growth is a random walk. This result is known as *"excess sensitivity"* of current consumption to lagged income.

Time Aggregation: Working (1960) is the first person to point out that *time aggregation can cause serial correlation* in aggregated data even if the disaggregated data follow a random walk. Then, from the point of view of PIH theory, excess sensitivity may be due to a spurious correlation.

To see this, suppose that consumption data were collected annually, so that time period t represent a year. But suppose that agents make decisions at a higher frequency, for example two times a year. Let τ index the period of time represented by a six-month duration, so that $\tau + (\tau + 1)$ index a full year. Then the measured change in annually collected consumption Δc_t^A is related to the underlying semi-annual, unmeasured consumption changes by:

$$\begin{aligned} \Delta c_t^A &= (c_\tau + c_{\tau+1}) - (c_{\tau-1} + c_{\tau-2}) \\ &= \Delta c_\tau + c_{\tau+1} - c_{\tau-2} \\ &= \Delta c_\tau + c_{\tau+1} - c_{\tau-2} + (\underbrace{\Delta c_{\tau+1} + \Delta c_\tau + \Delta c_{\tau-1} + c_{\tau-2} - c_{\tau+1}}_{=0}) \\ &= \Delta c_{\tau+1} + 2\Delta c_\tau + \Delta c_{\tau-1} \end{aligned}$$

Similarly for annual income change:

$$\Delta y_t^A = \Delta y_{\tau+1} + 2\Delta y_\tau + \Delta y_{\tau-1}$$

However, both Δc_t^A and Δy_t^A include terms that depend on $\tau-1$. But $\tau-1$ corresponds to the latter half of the previous year $t-1$. Therefore, it is possible that the measured correlation between Δc_t^A and Δy_{t-1}^A is entirely due to the overlapping time-span over which the data are collected, while the disaggregated consumption growth data Δc_τ are not correlated with disaggregated lagged income growth Δy_τ.

Notice, however, that c_t^A should then be uncorrelated with c_{t-2}^A and y_{t-2}^A, since these variables do not share a common six-month measurement period. Deaton (1992) estimates regression (10) by instrumenting Δy_{t-1} with Δc_{t-2} and Δy_{t-2} and finds:

$$\begin{array}{cccccc} \Delta c_t & = & 10.63 & + & 0.174 & \Delta y_{t-1} \\ & & (6.83) & & (2.18) & \end{array}$$

Although the significance is reduced, excess sensitivity is still present.

Predictable Income Change: Campbell and Mankiw (1989) consider a generalization of the PIH model. In their (reduced form) model, a fraction λ of households are "hand-to-mouth" (or as they call them, "rule of thumb") consumers who eat their income each period. If past income is a good predictor of future income, then excess sensitivity could be a result of a large fraction of hand-to-mouth consumers in the population. Their generalization generates a relationship of the form:

$$\Delta c_t = \lambda \Delta y_t + (1-\lambda)\varepsilon_t$$

where the PIH Euler equation is recovered if $\lambda = 0$. Campbell and Mankiw (1989) estimate the relationship by instrumenting for income change with lagged income growth, $\{\Delta y_{t-2-2}\}_{j=2}^{6}$. The first lag is $t-2$ due to the time-aggregation issue above. Campbell and Mankiw (1989) estimate:

$$\Delta c_t \quad = \quad \mu \quad + \quad 0.506 \quad \Delta y_t$$

where 0.506 is significantly different from zero at the 0.01% level. From the perspective of their model, $\lambda \approx 1/2$ can be interpreted as a resolution to the excess sensitivity puzzle *if* half of aggregate income is consumed by hand-to-mouth consumers.

Excess Smoothness: The "strict" Permanent Income Hypothesis mode has another puzzle when confronted with the data on consumption and income volatilities. Deaton (1987) first showed that, if income is serially correlated and persistent then the PIH predicts that the change in consumption should be more variable than the change in income. However, consumption is considerable smoother than income in the data. This is known as *excess smoothness* of the data relative to theory.

We now illustrate this point using the characterization of the Permanent Income Hypothesis model and the closed-form solutions we obtained in Section 2.3.

As a warmup, first suppose that that income follows a moving average representation:

$$y_t = \epsilon_t + \gamma \epsilon_{t-1}$$

Substituting this income process into optimal consumption dynamics in equation (8), we obtain:

$$\begin{aligned} \Delta c_t &= \frac{r}{1+r} \sum_{j=0}^{\infty} \left(\frac{1}{1+r} \right)^j \left(\mathbb{E}_t[y_{t+j}] - \mathbb{E}_{t-1}[y_{t+j}] \right) \\ &= \frac{r}{1+r} \left(\epsilon_t + \left(\frac{1}{1+r} \right) \gamma \epsilon_t \right) \end{aligned}$$

Assuming parameter values of $r = 0.04$ and $\gamma = (1+r)/2$, then the standard deviation of consumption change is:

$$\sigma(\Delta c) \;=\; \frac{r}{1+r} \left(1 + \frac{\gamma}{1+r} \right) \sigma(\epsilon) \;=\; \frac{0.04}{1.04}(1+0.5)\sigma(\epsilon) \;\approx\; 0.06\sigma(\epsilon)$$

Therefore if income follows a stationary process (featuring moderate serial correlation of $\gamma/(1+\gamma^2) \approx 0.4$), then consumption is smoother than income as in the data.[3]

Now consider a non-stationary process. Consider the class of difference stationary income representations, such as:

$$\Delta y_t = \alpha + \gamma \Delta y_{t-1} + \epsilon_t$$

To build intuition, let's first specialize to the case in which income is a martingale, $\alpha = \gamma = 0$:

$$\Delta y_t = \epsilon_t$$

so that $y_t = \mathbb{E}_t[y_{t+j}]$ for all $j \geq 0$. Then substituting the income process into optimal

[3]This result holds for a general ARMA process. See Chris Carroll's lecture notes for a proof.

consumption dynamics in equation (8), we obtain:

$$\begin{aligned}\Delta c_t &= \frac{r}{1+r}\sum_{j=0}^{\infty}\left(\frac{1}{1+r}\right)^j\Big(\mathbb{E}_t[y_{t+j}] - \mathbb{E}_{t-1}[y_{t+j}]\Big)\\ &= \frac{r}{1+r}\sum_{j=0}^{\infty}\left(\frac{1}{1+r}\right)^j\Big(y_t - y_{t-1}\Big)\\ &= \frac{r}{1+r}\sum_{j=0}^{\infty}\left(\frac{1}{1+r}\right)^j\epsilon_t\\ &= \epsilon_t\end{aligned}$$

This result tells us that if income follows a random walk, then consumption reactions one-to-one because it follows the same random walk. Intuitively, when income changes, the agent expects that the change is permanent and therefore adjusts consumption by the same amount as the income change.

Now consider a more general difference stationary process, in which $\mu_0 = 0$ but using US quarterly income data to estimate $\gamma = 0.26$. Also let $\Delta y_{t-1} = 0$ for simplicity. Then we can write the income process as:

$$\Delta y_t = \gamma \Delta y_{t-1} + \epsilon_t$$

which implies:

$$\begin{aligned}\Delta y_{t+j} &= \gamma\Delta y_{t+j-1} + \epsilon_{t+j}\\ &= \gamma^2\Delta y_{t+j-2} + \gamma\epsilon_{t+j-1} + \epsilon_{t+j}\\ &= \gamma^3\Delta y_{t+j-3} + \gamma^2\epsilon_{t+j-2} + \gamma\epsilon_{t+j-1} + \epsilon_{t+j}\\ &= \qquad\vdots \qquad\qquad \vdots \qquad\qquad \vdots\\ &= \gamma^{j-1}\underbrace{\Delta y_{t-1}}_{\equiv 0} + \sum_{i=0}^{j}\gamma^{j-i}\epsilon_{t+i}\end{aligned}$$

Therefore, taking expectations at period t:

$$\mathbb{E}_t[y_{t+j}] = \mathbb{E}_t\Big[y_{t+j-1} + \gamma\Delta y_{t+j-1} + \epsilon_{t+j}\Big]$$

$$
\begin{aligned}
&= \mathbb{E}_t[y_{t+j-1}] + \mathbb{E}_t\left[\sum_{i=0}^{j} \gamma^{j-i}\epsilon_{t+i}\right] \\
&= \mathbb{E}_t[y_{t+j-1}] + \mathbb{E}_t\left[\gamma^{j}\epsilon_{t}\right] \\
&= \mathbb{E}_t[y_{t+j-2}] + \mathbb{E}_t\left[\gamma^{j-1}\epsilon_{t}\right] + \mathbb{E}_t\left[\gamma^{j}\epsilon_{t}\right] \\
&= \qquad \vdots \qquad\qquad \vdots \qquad\qquad \vdots \\
&= \mathbb{E}_t\left[y_t + \sum_{i=1}^{j} \gamma^{i}\epsilon_{t}\right]
\end{aligned}
$$

And lastly note:

$$
\begin{aligned}
\mathbb{E}_t[y_t] - \mathbb{E}_{t-1}[y_t] &= \mathbb{E}_t[y_{t-1} + \gamma\Delta y_{t-1} + \epsilon_t] - \mathbb{E}_{t-1}[y_{t-1} + \gamma\Delta y_{t-1} + \epsilon_t] \\
&= \left(\underbrace{\mathbb{E}_t[y_{t-1} + \gamma\Delta y_{t-1}] - \mathbb{E}_{t-1}[y_{t-1} + \gamma\Delta y_{t-1} + \epsilon_t]}_{=0}\right) + \mathbb{E}_t[\epsilon_t] \\
&= \epsilon_t
\end{aligned}
$$

Again consider optimal consumption dynamics in equation (8):

$$
\begin{aligned}
\Delta c_t &= \frac{r}{1+r}\sum_{j=0}^{\infty}\left(\frac{1}{1+r}\right)^{j}\left(\mathbb{E}_t[y_{t+j}] - \mathbb{E}_{t-1}[y_{t+j}]\right) \\
&= \frac{r}{1+r}\sum_{j=0}^{\infty}\left(\frac{1}{1+r}\right)^{j}\left(\mathbb{E}_t\left[y_t + \sum_{i=1}^{j}\gamma^{i}\epsilon_t\right] - \mathbb{E}_{t-1}\left[y_t + \sum_{i=1}^{j}\gamma^{i}\epsilon_t\right]\right) \\
&= \frac{r}{1+r}\sum_{j=0}^{\infty}\left(\frac{1}{1+r}\right)^{j}\left(\epsilon_t + \mathbb{E}_t\left[\sum_{i=1}^{j}\gamma^{i}\epsilon_t\right] - \mathbb{E}_{t-1}\left[\sum_{i=1}^{j}\gamma^{i}\epsilon_t\right]\right) \\
&= \frac{r}{1+r}\sum_{j=0}^{\infty}\left(\frac{1}{1+r}\right)^{j}\left(1 + \sum_{i=1}^{j}\gamma^{i}\right)\epsilon_t \\
&= \frac{r}{1+r}\sum_{j=0}^{\infty}\left(\frac{1}{1+r}\right)^{j}\left(\sum_{i=0}^{j}\gamma^{i}\right)\epsilon_t \\
&= \frac{r}{1+r}\sum_{i=0}^{\infty}\left(\frac{\gamma}{1+r}\right)^{i}\sum_{j=0}^{\infty}\left(\frac{1}{1+r}\right)^{j}\epsilon_t
\end{aligned}
$$

$$\Delta c_t = \frac{1+r}{1+r-\gamma} \cdot \epsilon_t \tag{11}$$

where the last step holds since we happened to estimate $\gamma = 0.26 < 1$.

Finally, taking the standard deviation of consumption changes, we find that:

$$\sigma(\Delta c) = \frac{1+r}{1+r-\gamma}\sigma(\epsilon) = \frac{1.04}{1.04-0.26}\sigma(\epsilon) = 1.33\sigma(\epsilon)$$

Therefore, when income changes are positively serially correlated / sufficiently persistent, the Permanent Income Hypothesis predicts that consumption variability is *larger* than income variability. But this is a counterfactual prediction: the data show that consumption variability is actually *much smaller* than income variability.

Resolution: Campbell and Deaton (1989) consider the possibility that individuals in the data have a larger information set than the econometrician. It may be the case that consumption variability is not excessively low relative to income variability, but instead that the econometrician is estimating income variability with error.

Denote the time t information set of the individual $\mathcal{I}_t$ and that of the econometrician by $\Omega_t \subset \mathcal{I}_t$. Recall the characterization of optimal savings dynamics in equation (9), and notice that the individuals savings, denoted Δa^i_{t+1}, and the econometrician's prediction for individual savings, denoted Δa^e_{t+1}, are conditioned on their respective information sets:

$$\frac{1}{1+r}\Delta a^i_{t+1} = -\sum_{j=1}^{\infty}\left(\frac{1}{1+r}\right)^j \mathbb{E}_t[\Delta y_{t+j} \mid \mathcal{I}_t]$$

$$\frac{1}{1+r}\Delta a^e_{t+1} = -\sum_{j=1}^{\infty}\left(\frac{1}{1+r}\right)^j \mathbb{E}_t[\Delta y_{t+j} \mid \Omega_t]$$

Then the econometrician's prediction error is:

$$\frac{1}{1+r}\left(\Delta a^i_{t+1} - \Delta a^e_{t+1}\right) = -\sum_{j=1}^{\infty}\left(\frac{1}{1+r}\right)^j \left(\mathbb{E}_t[\Delta y_{t+j} \mid \Omega_t] - \mathbb{E}_t[\Delta y_{t+j} \mid \mathcal{I}_t]\right)$$

To the extent that the econometrician makes errors in prediction, the savings rate then contains useful information for identifying individual expectations of income growth. Therefore the savings rate can provide additional forecasting power. But then the Hall (1978) and Flavin (1981) specifications could be augmented with savings information. Consider the following vector autoregression:

$$\begin{pmatrix} \Delta y_t \\ \frac{1}{1+r}\Delta a_{t+1} \end{pmatrix} = \begin{pmatrix} \zeta_{11} & \zeta_{12} \\ \zeta_{21} & \zeta_{22} \end{pmatrix} \begin{pmatrix} \Delta y_{t-1} \\ \frac{1}{1+r}\Delta a_t \end{pmatrix} + \begin{pmatrix} u_{1t} \\ u_{2t} \end{pmatrix}$$

or rewritten more parsimoniously:

$$x_t = Ax_{t-1} + u_t$$

Due to linearity of the system of equations, we can write j-step ahead forecasts as:

$$\mathbb{E}_t[x_{t+j}] = A^j x_t$$

which can be rewritten in terms of the underlying system:

$$\mathbb{E}_t[\Delta y_{t+j}] = e_1 A^j x_{t+j}$$

$$\frac{1}{1+r}\Delta a_{t+1} = -\sum_{j=1}^{\infty}\left(\frac{1}{1+r}\right)^j e_1 A^j x_t \tag{12}$$

where $e_1 = (1, 0)$ and $e_2 = (0, 1)$ are 1×2 vectors that pick out the income and savings equations, respectively. In other words:

$$e_1 x_t = \Delta y_t$$

$$e_2 x_t = \frac{1}{1+r}\Delta a_{t+1}$$

Note that the linearity of the system of equations allows us to rewrite equation (12) by dividing through by x_t:

$$e_2 = -e_1 \sum_{j=1}^{\infty}\left(\frac{1}{1+r}\right)^j A^j$$

It turns out this places a restriction on the parameters in A. To see this, if we require that $|A| < 1$ then the matrix geometric series summations are:

$$\sum_{j=0}^{\infty}\left(\frac{A}{1+r}\right)^j = \left(I - \frac{A}{1+r}\right)^{-1}$$

$$\sum_{j=1}^{\infty}\left(\frac{A}{1+r}\right)^j = \left(I - \frac{A}{1+r}\right)^{-1} - I$$

where I is a 2×2 identify matrix. Substitute into equation (2.4) and manipulate:

$$
\begin{aligned}
e_2 &= -e_1\left(\left(I - \frac{A}{1+r}\right)^{-1} - I\right) \\
e_2\left(I - \frac{A}{1+r}\right) &= e_1\left(\left(I - \frac{A}{1+r}\right) - I\right) \\
(1+r)e_2 &= (e_2 - e_1)\, A \\
(1+r)e_2 &= (-1, 1)A \\
\Big(0, (1+r)\Big) &= \Big(-\zeta_{11} + \zeta_{21}\ ,\ -\zeta_{12} + \zeta_{22}\Big)
\end{aligned}
$$

Therefore, we have derived the following parameter restrictions:

$$
\begin{aligned}
\zeta_{21} &= \zeta_{11} \\
\zeta_{22} &= \zeta_{12} + (1+r)
\end{aligned} \tag{13}
$$

Notice that these restrictions have implications for consumption changes. To see this, difference the budget constraint and substitute the restrictions in (13):

$$
\begin{aligned}
\Delta c_t &= \Delta y_t + \Delta a_t - \frac{1}{1+r}\Delta a_{t+1} \\
&= \left(\zeta_{11}\Delta y_{t-1} + \zeta_{12}\frac{1}{1+r}\Delta a_t + u_{1t}\right) + \Delta a_t - \left(\zeta_{21}\Delta y_{t-1} + \zeta_{22}\frac{1}{1+r}\Delta a_t + u_{2t}\right) \\
&= \left(\zeta_{11}\Delta y_{t-1} + \zeta_{12}\frac{1}{1+r}\Delta a_t + u_{1t}\right) + \Delta a_t - \left(\zeta_{11}\Delta y_{t-1} + (\zeta_{12} + 1 + r)\frac{1}{1+r}\Delta a_t + u_{2t}\right) \\
&= u_{1t} - u_{2t}
\end{aligned}
$$

Therefore, consumption follows a random walk. This means that under these parameter restrictions, consumption is not predictable by lagged variables such as savings or income change.

Campbell and Deaton (1989) estimate the vector autoregression and, in several different cases, **reject** the restriction that $\zeta_{11} = \zeta_{21}$. As a result, they consider the following alternative specification. Define an "excess sensitivity" parameter, denoted χ. Suppose that ζ_{21} instead equaled $\zeta_{11} - \chi$ so that:

$$A = \begin{pmatrix} \zeta_{11} & \zeta_{12} \\ \zeta_{11} - \chi & \zeta_{12} + 1 + r \end{pmatrix}$$

The parameter restriction $\zeta_{21} = \zeta_{11}$ in (13) changes to $\zeta_{21} = \zeta_{11} + \chi$ and therefore consumption changes now depend on lagged income changes:

$$\Delta c_t = \chi \Delta y_{t-1} + (u_{1t} - u_{2t}) \tag{14}$$

The results shows that consumption responds positively to lagged income changes. Furthermore, since lagged income changes predict future income changes, $\chi > 0$ can be interpreted as a response of consumption to anticipated changes in current income.

Now let's see what assuming some positive amount of "excess sensitivity" implies for "excess smoothness." Let's derive consumption using the iterative technique that we developed in equation (11). However, because we are now considering a bivariate system, the derivation is much more tedious. Stick with me!

First, consider the income innovation at time t:

$$\begin{aligned}
\mathbb{E}_t[y_t] - \mathbb{E}_{t-1}[y_t] &= \mathbb{E}_t[y_{t-1} + \zeta_{11}\Delta y_{t-1} + \zeta_{12}\Delta a_t + u_{1t}] - \mathbb{E}_{t-1}[y_{t-1} + \zeta_{11}\Delta y_{t-1} + \zeta_{12}\Delta a_t + u_{1t}] \\
&= \Big(\underbrace{\mathbb{E}_t[y_{t-1} + \zeta_{11}\Delta y_{t-1} + \zeta_{12}\Delta a_t] - \mathbb{E}_{t-1}[y_{t-1} + \zeta_{11}\Delta y_{t-1} + \zeta_{12}\Delta a_t]}_{=0} \Big) + u_{1t} \\
&= u_{1t}
\end{aligned}$$

It will be useful to re-derive this result using our vector notation (e_1, e_2, A, x, u):

$$\begin{aligned}
\mathbb{E}_t[y_t] - \mathbb{E}_{t-1}[y_t] &= \mathbb{E}_t[y_{t-1} + e_1(Ax_{t-1} + u_t)] - \mathbb{E}_{t-1}[y_{t-1} + e_1(Ax_{t-1} + u_t)] \\
&= y_{t-1} + e_1(Ax_{t-1} + u_t) - y_{t-1} + e_1 A x_{t-1} \\
&= e_1 u_t
\end{aligned}$$

Iterate one step ahead to $t+1$, still using vector notation:

$$\begin{aligned}
\mathbb{E}_t[y_{t+1}] - \mathbb{E}_{t-1}[y_{t+1}] &= \mathbb{E}_t[y_t + e_1(Ax_t + u_{t+1})] - \mathbb{E}_{t-1}[y_t + e_1(Ax_t + u_{t+1})] \\
&= (\mathbb{E}_t[y_t] - \mathbb{E}_{t-1}[y_t]) + (\mathbb{E}_t[e_1 A x_t] - \mathbb{E}_{t-1}[e_1 A x_t])
\end{aligned}$$

$$= (e_1 u_t) + (\mathbb{E}_t[e_1 A(Ax_{t-1} + u_t)] - \mathbb{E}_{t-1}[e_1 A(Ax_{t-1} + u_t)])$$

$$= (e_1 u_t) + (e_1 A u_t)$$

$$= e_1(1 + A)u_t$$

Iterate one more step ahead to $t + 2$:

$$\mathbb{E}_t[y_{t+2}] - \mathbb{E}_{t-1}[y_{t+2}] = \mathbb{E}_t[y_{t+1} + e_1(Ax_{t+1} + u_{t+2})] - \mathbb{E}_{t-1}[y_{t+1} + e_1(Ax_{t+1} + u_{t+2})]$$

$$= (\mathbb{E}_t[y_{t+1}] - \mathbb{E}_{t-1}[y_{t+1}]) + (\mathbb{E}_t[e_1 A x_{t+1}] - \mathbb{E}_{t-1}[e_1 A x_{t+1}])$$

$$= (e_1(1 + A)u_t) + (\mathbb{E}_t[e_1 A^2 x_t] - \mathbb{E}_{t-1}[e_1 A^2 x_t])$$

$$= (e_1(1 + A)u_t) + (\mathbb{E}_t[e_1 A^2(Ax_{t-1} + u_t)] - \mathbb{E}_{t-1}[e_1 A^2(Ax_{t-1} + u_t)])$$

$$= (e_1(1 + A)u_t) + (e_1 A^2 u_t)$$

$$= e_1(1 + A +^2)u_t$$

It should be clear by now that forward iteration will yield:

$$\mathbb{E}_t[y_{t+\tau}] - \mathbb{E}_{t-1}[y_{t+\tau}] = \sum_{j=0}^{\tau} e_1 A^j u_t$$

Therefore, we can derive the optimal change in consumption in response to unanticipated changes in permanent income, as we did in equation (11):

$$\Delta c_t = \frac{r}{1+r} \sum_{i=0}^{\infty} e_1 \left(\frac{A}{1+r}\right)^i u_t \sum_{j=0}^{\infty} \left(\frac{1}{1+r}\right)^j$$

$$= e_1 \left(I - \frac{A}{1+r}\right)^{-1} u_t$$

$$= \frac{1}{\left(1 - \frac{\zeta_{11}}{1+r}\right)\left(\frac{-\zeta_{12}}{1+r}\right) - \left(\frac{\zeta_{11}-\chi}{1+r}\right)\left(\frac{\zeta_{12}}{1+r}\right)} e_1 \begin{pmatrix} -\frac{\zeta_{12}}{1+r} & \frac{\zeta_{12}}{1+r} \\ \frac{\zeta_{11}-\chi}{1+r} & 1 - \frac{\zeta_{11}}{1+r} \end{pmatrix} u_t$$

$$= \frac{-\frac{\zeta_{12}}{1+r} u_{1t} + \frac{\zeta_{12}}{1+r} u_{2t}}{\left(1 - \frac{\zeta_{11}}{1+r}\right)\left(\frac{-\zeta_{12}}{1+r}\right) - \left(\frac{\zeta_{11}-\chi}{1+r}\right)\left(\frac{\zeta_{12}}{1+r}\right)}$$

$$= \frac{u_{1t} - u_{2t}}{\left(1 - \frac{\zeta_{11}}{1+r}\right) + \left(\frac{\zeta_{11}-\chi}{1+r}\right)}$$

$$\Delta c_t = \frac{u_{1t} - u_{2t}}{1 - \frac{\chi}{1+r}}$$

We see that when $\chi = 0$, we recover that consumption equals permanent income and both are a martingale given by $u_{1t} - u_{2t}$. However, as $\chi \to 1+r$, consumption may have an arbitrarily large response to unanticipated changes in income. That is, *when there is excess sensitivity to lagged income* $(\chi > 0)$, *consumption responses to unanticipated changes in income are amplified*. Therefore, Campbell and Deaton (1989) conclude that "There is no contradiction between excess sensitivity and excess smoothness; they are the same phenomenon."

A note on the deeper mechanism at play. The excess sensitivity parameter χ operates through the effect that lagged income has on savings, which in turn is *is internalized by the individual agent in his forecast of discounted expected income change*. Put differently, when $\chi > 0$, the change in permanent income reflects dynamic changes in savings that are due to changes in income. If past savings predicts income, then savings behavior affects permanent income.

3 Friedman / Buffer Stock Model

We ended the last section by resolving the excess sensitivity and excess smoothness puzzles by attributing dependence of consumption changes to bias: the econometrician does not perfectly observe the agent's information set. We begin this section by discussing a second resolution: precautionary savings through borrowing constraints or prudence in utility. As an alternative to the "strict" Permanent Income Hypothesis, which does not feature a precautionary motive, we consider the Buffer Stock model, which does feature a precautionary motive. We develop the model's theoretical implications and empirically evaluate them.

3.1 Precautionary Savings

Borrowing Constraints: Parker (1999), Souleles (1999) and Johnson et al. (2006) show that consumption is very sensitive to anticipated changes in income through governmental transfers (tax rebates, social security changes, etc.). These studies find that the consumption response is stronger for households with low liquid wealth and low income, which suggests households face borrowing constraints that impede optimal consumption smoothing.

Consider "strict" Permanent Income Hypothesis model once more. Suppose that agents face a borrowing constraint that forbids borrowing (a so called, *no borrowing constraint*) given by $a_{t+1} \geq 0$. Then optimal consumption now depends on the borrowing constraint. If the borrowing constraint binds ($a_{t+1} = 0$), consumption no longer follows a random walk and is instead determined by the budget constraint:

$$c_t = y_t + a_t - \underbrace{\frac{1}{1+r} a_{t+1}}_{=0}$$

Optimal consumption, therefore, takes the form:

$$c_t = \left\{ \begin{array}{ll} \mathbb{E}_t[c_{t+1}] & \text{if } a_{t+1} > 0 \\ y_t + a_t & \text{if } a_{t+1} = 0 \end{array} \right\}$$

Suppose that we consider a particular stochastic process for income. Suppose $y_t = y_{t-1} + \varepsilon_t$ with $\mathbb{E}_t[\varepsilon_t] = 0$ (e.g. permanent shocks). Now consider optimal changes in savings in equation (9). Because $\Delta y_{t+j} = \varepsilon_{t+j}$ for all $j \geq 0$, we know that $\Delta a_{t+1} = 0$. Therefore savings never changes and all innovations to income are consumed.

Now suppose that y_t is an iid random variable with $\mathbb{E}_t[y_{t+j}] = 0$ for all $j > 0$ (e.g.transitory shocks only). Then $\Delta y_{t+j} = \varepsilon_{t+j} - \varepsilon_{t-1+j}$ and:

$$\Delta a_{t+1} = -\sum_{j=1}^{\infty} \mathbb{E}_t[\varepsilon_{t+j} - \varepsilon_{t-1+j}] = \varepsilon_t$$

But then changes in savings follows a random walk and any finite borrowing constraint will bind with probability one at $t \to \infty$.

Now iterate forward on the optimal consumption decision, to obtain:

$$\begin{aligned} c_t &= \min\{y_t + a_t, \mathbb{E}_t[c_{t+1}]\} \\ &= \min\{y_t + a_t, \mathbb{E}_t[\min\{y_{t+1} + a_{t+1}, \mathbb{E}_{t+1}[c_{t+2}]\}]\} \\ &= \dots \end{aligned}$$

If the variance of the income shock increases, then the probability of receiving a sufficiently negative income shock also increases. Supposing the agent is unconstrained at time t, then his consumption is:

$$c_t = \mathbb{E}_t[\min\{y_{t+1} + a_{t+1}, \mathbb{E}_{t+1}[c_{t+2}]\}]$$

But if the variance of y increases, the probability that future constraints bind increases and reduces consumption at time t. Therefore, when shocks are transitory the presence of borrowing constraints can disrupt consumption smoothing unless agents forgo consumption today and increase savings for tomorrow. In other words, *liquidity constraints generate incentives to accumulate precautionary savings.*

Prudence: Another mechanism by which agents may be sensitive to income variability is an aversion to consumption variability. If a consumer has preferences for smooth consumption profiles, then greater income variability implies an incentive to accumulate savings to self-insure against low income realizations.

Now assume, contrary to quadratic utility, that agents have a utility function $u : \mathbb{R}_+ \to \mathbb{R}$ that is strictly increasing, $u'(c) > 0$, and strictly concave, $u''(c) < 0$. Recall that the Arrow-Pratt measure, or coefficient of absolute risk aversion is:

$$A(c) \equiv -\frac{u''(c)}{u'(c)}$$

Therefore decreasing absolute risk aversion (DARA) requires:

$$A'(c) = -\frac{u'''(c)}{u'(c)} + \left(\frac{u''(c)}{u'(c)}\right)^2 < 0$$

Rewritten as a condition on the third derivative of the utility function and noting $u'(c) > 0$ and $u''(c) < 0$, we obtain:

$$u'''(c) > \frac{u''(c)^2}{u'(c)} > 0$$

Following Kimball (1990), the convexity of the marginal utility function, e.g. $u'''(c) > 0$, is called *prudence*.[4] As we will show momentarily, this property implies that increases in uncertainty over future income induce individuals to consume less today and save more for tomorrow.

[4]Analogously to the Arrow-Pratt measure of absolute risk aversion, we can write Kimball's measure of absolute prudence:

$$\alpha(c) \equiv -\frac{u'''(c)}{u''(c)}$$

which implies that a utility function exhibits prudence when $\alpha(c) > 0$ and $A(c) > 0$. A CRRA utility function satisfies these conditions.

Following Leland (1968), we will illustrate how the class of DARA utility functions generate a precautionary savings motive. Suppose a two-period economy with $t = 0, 1$. An agent has an instantaneous strictly increasing and strictly concave utility function. Suppose the agent is endowed with y_0 units of income, but at time $t = 1$ receives an income shock $y_1 \sim F(y_1)$. The agent can consume and save each period (but clearly has no incentive to save in the terminal period, so $a_2 = 0$). The agent's problem is:

$$v(y_0) = \max_{c_0, c_1, a_1} u(c_0) + \beta \mathbb{E}_0[u(c_1)]$$

$$\text{s.t.} \quad c_0 + a_1 \leq y_0$$

$$c_1 \leq y_1 + (1+r)a_1$$

Assuming still (without loss of generality) that $\beta(1+r) = 1$, the Euler equation is:

$$u'(y_0 - a_1) = \mathbb{E}[u'((1+r)a_1 + y_1)]$$

Then the left hand side is strictly increasing in a_1 and the right hand side is strictly decreasing in a_1, by the concavity of the utility function ($u'' < 0$). Therefore we can find a value a_1^* satisfying the Euler equation given the endowment y_0. From the budget constraint, $c_0^* = y_0 - a_1^*$.

Now suppose that the variance of the distribution $F(y)$ increases in a mean-preserving fashion (e.g. suppose $y = \bar{y} + \varepsilon$ with $\mathbb{E}[\varepsilon] = 0$ and $\mathbb{E}[\varepsilon^2] = \sigma_\varepsilon^2$, and increase σ_ε). Since $u(c)$ was chosen from the DARA class, we know that $u'(c)$ is a convex function since $u'''(c) > 0$. Then by Jensen's inequality the right hand side of the Euler equation increases with uncertainty over y_1. That is, the expected marginal utility of consumption increases with uncertainty over future income, precisely because high marginal utility (e.g. low y) outcomes become more probable. Then, the value of a_1^* satisfying the Euler equation must increase, which tightens the time 0 budget constraint and induces a decrease in c_0^*. Therefore, higher uncertainty over tomorrow's income increases the agent's precautionary motive for savings due.

Lastly, note the above argument goes through for a finite horizon economy. Suppose that there are T periods and each period the agent draws an iid income shock. The agent's dynamic program is:

$$v_t(x_t) = \max_{c_t, x_{t+1}} u(c_t) + \beta \mathbb{E}[v_{t+1}(x_{t+1})]$$

$$\text{s.t.} \quad x_{t+1} = (1+r)(x_t - c_t) + y_{t+1}$$

where x_t is called *cash in hand* and is a convenient state variable when shocks are iid. To see the equivalence with the previous budget constraints, write the following:

$$c_t + a_{t+1} = y_t + (1+r)a_t \equiv x_t$$

$$a_{t+1} = x_t - c_t$$

$$x_{t+1} \equiv (1+r)a_{t+1} + y_{t+1} = (1+r)(x_t - c_t) + y_{t+1}$$

$$x_{t+1} = (1+r)(x_t - c_t) + y_{t+1}$$

Now, taking first order conditions we arrive at the following Euler equation:

$$u'(c_t) = \beta(1+r)\mathbb{E}\left[v'_{t+1}\Big((1+r)(x_t - c_t) + y_{t+1}\Big)\right]$$

If $v'''_{t+1}(x) > 0$, then the previous argument applies and increases in uncertainty about y_{t+1} induce a precautionary savings motive. To show that $v'''_{t+1}(x) > 0$, simply notice that at period $T+1$ the value function is $v_{T+1}(x) = 0$ and at time T:

$$v_T(x_T) = \max_{c_T}\{u(c_T) \text{ s.t. } c_T \leq x_T\}$$

But then if $u'''(c) > 0$ and $v_T(x) = u(x)$ it must be the case that $v'''_T(x) > 0$. By backward induction, all previous value functions inherit prudence, $v'''_t(x) > 0$ for all $t < T$.

3.2 Patience and Buffer Stock Savings

Let's now return to the canonical consumption-savings problem we stated in section 2.2. Recall that the infinite-horizon, recursive consumption-savings problem is:

$$v(a_t, y_t) = \max_{c_t, a_{t+1}} u(c_t) + \beta\mathbb{E}_t\left[v(a_{t+1}, y_{t+1})\right]$$

$$\begin{aligned} \text{s.t.} \quad c_t + a_{t+1} &\leq y_t + (1+r)a_t \\ a_{t+1} &\geq \underline{a}_{t+1} \\ c_t &\geq 0 \end{aligned}$$

where the Lagrange multiplier on the budget constraint is denoted λ_t and the Lagrange multiplier on the borrowing constraint is denoted μ_t. Assume that the utility function satisfies strict concavity and prudence, such as the CRRA utility function. The Euler equation is:

$$u'(c_t) \geq \beta(1+r)\mathbb{E}_t[u'(c_{t+1})]$$

which holds with equality when the borrowing constraint is slack.

In this section we want to study the effect of $\beta(1+r)$ on agents' optimal consumption. Intuitively, $\beta(1+r)$ determines the agent's desire to intertemporally substitute consumption. If $\beta(1+r) \geq 1$ then the agent can be thought of as *patient*: the agent prefers to consume in the future. If $\beta(1+r) < 1$ then the agent can be thought of as *impatient*: the agent prefers to consume today.

We want to show that under certain assumptions on the stochastic process for income and on agents' intertemporal substitution motives given by $\beta(1+r)$, there exists an endogenously determined upper bound on savings, $\bar{a} < \infty$, below which agents accumulate savings and above which run down savings. We start with assuming deterministic income paths and then rederive results with stochastic income.

While I provide informal justification for results, formal proofs are contained in Ljungqvist and Sargent's (2012) chapter on "Self-Insurance" as well as papers by Schechtman and Escudero (1977), Chamberlain and Wilson (2000) and Carroll (2012).

3.2.1 Deterministic Income

Case of $\beta(1+r) > 1$: When income is deterministic, the Euler equation can be written:

$$u'(c_t) \geq \beta(1+r)u'(c_{t+1})$$

If $\beta(1+r) > 1$ then:

$$u'(c_t) \geq \beta(1+r)u'(c_{t+1}) > u'(c_{t+1})$$

By the concavity of $u(c)$: $c_{t+1} > c_t$. The consumption allocation grows indefinitely and diverges to infinity as $t \to \infty$. Therefore, in the absence of income uncertainty, $\beta(1+r) > 1$ provides the agent with too high of a savings incentive and the postpones consumption to the infinite horizon.

Case of $\beta(1+r) = 1$: If the borrowing constraint never binds, then the Euler equation immediately informs us that $u'(c_t) = u'(c_{t+1})$ or $c_t = c_{t+1}$. In this case, the agent would perfectly smooth consumption. If the borrowing constraint binds in some period t, then the Euler equation informs us that $c_{t+1} > c_t$, that the agent is impatient and would like the transfer consumption from future periods to today but cannot. However, an occasionally binding borrowing constraint with an income path that is bounded above by some $\bar{y} < \infty$ is sufficient to guarantee perfect consumption smoothing after some date.

Intuitively, we can find a date for which human wealth (previously defined) achieves is maximum:

$$\tau \equiv \arg\sup_t \left\{ \frac{r}{1+r} \sum_{j=0}^{\infty} \left(\frac{1}{1+r} \right)^j y_{t+j} \right\}$$

which is well defined if $\sup_{j>0}\{y_{t+j}\} < \infty$ for all t. Before such a date τ consumption will grow, reflecting the impact of borrowing constraints. Agents want to borrow against their increasing future human wealth. After τ, human wealth is high and will decline. The agent will accumulate a sufficiently large stock of savings to keep consumption constant as income decreases. When human wealth is decreasing, the agent does not wish to borrow. Therefore, consumption and savings converge in this case.

Case of $\beta(1+r) < 1$: We will show that after some period τ, the agent perfectly smooths consumption when $\beta(1+r) < 1$ as well. To simplify (although the arguments go through with minor modification otherwise) let $y_t = \bar{y}$ for all t. Furthermore, consider the no borrowing constraint $\underline{a}_t = 0$ for all t.

There are two cases to consider. The trivial case is when at time t we have a binding borrowing constraint so that $a_{t+1} = 0$ and $c_{t+1} = \bar{y}$. Then for all $j > 0$ we will also have $a_{t+1+j} = 0$ and $c_{t+j} = \bar{y}$. To see this, write the sequence of budget constraints:

$$c_t + 0 = \bar{y} + (1+r)a_t$$

$$c_{t+1} + 0 = \bar{y} + 0$$

$$c_{t+2} + 0 = \bar{y} + 0$$

Next consider the case in which $a_{t+1} > 0$ at time t. It will be useful to reformulate the problem using cash-in-hand as a state variable (as in last section 3.1). We can prove the following lemma immediately.

1. If $a_{t+1} > 0$ then the Euler equation (first line below) combined with the envelope

condition (second line below) imply that cash-in-hand is a decreasing sequence:

$$u'(c_t) = \beta(1+r)v'(x_{t+1})$$
$$v'(x_t) = \beta(1+r)v'(x_{t+1})$$
$$v'(x_t) < v'(x_{t+1})$$
$$x_t > x_{t+1}$$

where the last two lines follow from $\beta(1+r) < 1$ and the concavity of the value function.

Given the lemma, we will show that if at time t cash-in-hand convergences to $\bar{y}$ from above, then for all $j > 0$ we will again have $a_{t+1+j} = 0$ and $c_{t+j} = \bar{y}$. Prove this by contradiction. Suppose instead that $x_t = \bar{y}$ but $a_{t+1} > 0$. Then by the Euler equation (first line), envelope condition (second line), impatience (third line) and concavity of the value function (last two lines):

$$u'(c_t) = \beta(1+r)v'(x_{t+1})$$
$$v'(\bar{y}) = \beta(1+r)v'((1+r)a_{t+1} + \bar{y})$$
$$v'(\bar{y}) < v'((1+r)a_{t+1} + \bar{y})$$
$$v'(\bar{y}) < v'(\bar{y})$$
$$\bar{y} > \bar{y}$$

But then $\bar{y} > \bar{y}$ is a contradiction. $\rightarrow\leftarrow$

3.2.2 Stochastic Income

With stochastic income, the precautionary motive is now active. Furthermore, the precautionary motive may interact with borrowing constraints.

Doob's Martingale Convergence Theorem (MCT): We will use the following theorem heavily in this section. Define the variable:

$$M_t \equiv \big(\beta(1+r)\big)^t u'(c_t)$$

and rewrite the Euler equation in the following equivalent form:

$$\Big(\beta(1+r)\Big)^t u'(c_t) \geq \Big(\beta(1+r)\Big)^{t+1} \mathbb{E}_t[u'(c_{t+1})]$$

$$M_t \geq \mathbb{E}_t[M_{t+1}]$$

The reformulation shows that M_t follows a *super martingale*. Given that $M_t > 0$ for all t, Doob's MCT guarantees that M_t converges almost surely to a finite limit, $\lim_{t\to\infty} M_t = \bar{M} < \infty$.

Case of $\beta(1+r) > 1$: As in the deterministic case, consumption will diverge in the limit. If $\beta(1+r) > 1$ then $\beta^t(1+r)^t$ converges to infinity as $t \to \infty$. But then for Doob's MCT to hold, it must be the case that $u'(c_t)$ decrease to offset increases arising from $\beta(1+r) > 1$. That is, in order to maintain M_t finite, $u'(c_t)$ must decrease as $(\beta(1+r))^t$ increases. Since the utility function is concave, this requires that c_t increases. Therefore $c_t \to \infty$ as $t \to \infty$ since $\lim_{t\to\infty}(\beta(1+r))^t = \infty$. Also as before, savings must diverge to finance the postponement of consumption into the infinite future.

Case of $\beta(1+r) = 1$: In the stochastic case, unlike the deterministic income case, assuming $\beta(1+r) = 1$ generates a divergent consumption path. To see this, consider the Euler equation when the utility function exhibits prudence, $u'''(c) > 0$:

$$u'(c_t) \geq \mathbb{E}_t[u'(c_{t+1})]$$

$$u'(c_t) > u'(E_t[c_{t+1}])$$

$$c_t < E_t[c_{t+1}]$$

where the second inequality is due to prudence and Jensen's inequality, and the last inequality is due to the concavity of the utility function. But then consumption is increasing over time and diverges in the limit.

Case of $\beta(1+r) < 1$: We will show that if y is iid and lies on a bounded interval $[\underline{y}, \bar{y}]$ with $\bar{y} > \underline{y}$ then there exists an endogenous upper bound x^* on cash-in-hand such that if $x_t > x^*$ then the agent optimally chooses $\bar{x}_{t+1} < x^*$ where $\bar{x}_{t+1}$ is cash-in-hand when $y_{t+1} = \bar{y}$. If cash-in-hand is bounded then so is consumption.

First prove the follow three lemmas, relying heavily on the precautionary motives derived from restrictions on utility classes:

1. *Lemma 1:* If $u''(c) < 0$, then consumption is increasing with cash-in-hand,

$$\frac{\partial c_t}{\partial x_t} > 0$$

Proof: From the envelope condition, $u'(c_t) = v'(x_t)$, where the value function inherits the concavity of the utility function. Then:

$$v''(x_t) = u''(c_t)\frac{\partial c_t}{\partial x_t}$$

$$\frac{\partial c_t}{\partial x_t} = \frac{v''(x_t)}{u''(c_t)} > 0$$

where the last inequality (> 0) follows from the concavity of both the utility and value functions. ■

2. *Lemma 2 [Carroll and Kimball (1996)]:* Fix a parameter, $\kappa > 1$. If $u(c)$ is in the DARA class, such that:

$$\frac{u'''(c)}{u''(c)} = \kappa\frac{u''(c)}{u'(c)}$$

and $v(x)$ satisfies:

$$\frac{v'''(x)}{v''(x)} \leq \kappa\frac{v''(x)}{v'(x)}$$

then the consumption policy function $c(x)$ is concave.

Proof: If we define $x(c)$ as the cash-in-hand associated with a level c of consumption then we can write savings as $s(c) = x(c) - c$. If $x(c)$ is convex then it must be the case that $c(x)$ is concave. But if $x(c)$ is convex, then so is savings: $s''(c) = x''(c) \geq 0$. Therefore we will show that $s(c)$ is a convex function.

Define the discounted expected value of utility as:

$$\phi(x - c) \equiv \beta\mathbb{E}[v((1 + r)(x - c) + \varepsilon)]$$

Then the right hand side of the Euler equation is:

$$\phi'(x - c) = \beta(1 + r)\mathbb{E}[v'((1 + r)(x - c) + \varepsilon')]$$

Denoting the Lagrange multiplier on the budget constraint as λ, we know that $\lambda = u'(c) = \phi'(x - c)$. Define $f(\lambda) \equiv u'^{-1}(c)$ and $g(\lambda) \equiv \phi'^{-1}(\lambda)$.

Using the facts that $\lambda = f^{-1}(c)$ and $x - c = g(\lambda)$:

$$s(c) = g(\lambda) = g(f^{-1}(c))$$

Taking derivatives we obtain:[5]

$$\begin{aligned}
s'(c) &= g'(f^{-1}(c))\frac{\partial f^{-1}(c)}{\partial c} = \frac{g'(f^{-1}(c))}{f'(f^{-1}(c))} \\
s''(c) &= \frac{g'(f^{-1}(c))}{f'(f^{-1}(c))^2} \cdot \left(\frac{g''(f^{-1}(c))}{g'(f^{-1}(c))} - \frac{f''(f^{-1}(c))}{g'(f^{-1}(c))} \right) \\
&= \frac{g'(\lambda)}{\lambda f'(\lambda)^2} \cdot \left(\frac{\lambda g''(\lambda)}{g'(\lambda)} - \frac{\lambda f''(\lambda)}{g'(\lambda)} \right) \\
&= \frac{1/\phi''(s(c))}{u'(c)/u''(c)^2} \cdot \left(\frac{-\phi'(s(c))\phi'''(s(c))/\phi''(s(c))^3}{1/\phi''(s(c))} - \frac{-u'(c)u'''(c)/u''(c)^3}{1/u''(c)} \right) \\
&= \underbrace{\frac{1}{\phi''(s(c))}}_{<0} \underbrace{\frac{u''(c)^2}{u'(c)}}_{>0} \cdot \left(\underbrace{\frac{u'(c)u'''(c)}{u''(c)^2}}_{=\kappa} - \underbrace{\frac{\phi'(s(c))\phi'''(s(c))}{\phi''(s(c))^2}}_{\geq \kappa} \right) \\
&\geq 0
\end{aligned}$$

where $\phi(s(c))$ inherits positive monotonicity and concavity from the value function, as well as inherits the assumed property that

$$\frac{v'''(x)}{v''(x)} \leq \kappa \frac{v''(x)}{v'(x)} \quad \Longrightarrow \quad \frac{\phi'(s(c))\phi'''(s(c))}{\phi''(s(c))^2} \geq \kappa$$

We have shown that the optimal savings policy function is convex and therefore the consumption policy function must be concave. To prove strict concavity, see Carroll and Kimball (1996), Corollary 1. ■

[5] Define two functions $g, f : \mathbb{R} \to \mathbb{R}$. Let $y = f(x)$ and $g(f(x)) = x$ so that g is the inverse of f. Recall that derivatives of inverse functions are given by:

$$g'(f(x))f'(x) = 1 \quad \Longrightarrow \quad \frac{\partial}{\partial y} f^{-1}(y) = \frac{1}{f'(f^{-1}(y))}$$

And for the second derivative:

$$g''(f(x))f'(x)^2 + g'(f(x))f''(x) = 0 \quad \Longrightarrow \quad \frac{\partial^2}{\partial y^2} f^{-1}(y) = -\frac{f''(f^{-1}(y))}{f'(f^{-1}(y))^3}$$

3. *Lemma 3:* If $u(c)$ is in the DARA class, then

$$\lim_{x\to\infty} \frac{u'(c(x_{t+1}))}{u'(c(\bar{x}_{t+1}))} = 1$$

Proof: Take a first order Taylor approximation of $u'(c(x))$ around the point $x = \bar{x}$:

$$u'(c(x)) \approx u'(c(\bar{x})) + u''(c(\bar{x}))c'(\bar{x})(x - \bar{x})$$

where $c'(x)$ is the first derivative of the consumption policy function with respect to cash-in-hand. Now take expectations:

$$\mathbb{E}[u'(c(x))] \approx u'(c(\bar{x})) + u''(c(\bar{x}))c'(\bar{x})\mathbb{E}[y' - \bar{y}]$$

where $x' - \bar{x}' = (1+r)x + y' - (1+r)x - \bar{y} = y' - \bar{y}$. Divide by $u'(c(\bar{x}'))$:

$$\frac{\mathbb{E}[u'(c(x))]}{u'(c(\bar{x}))} \approx 1 - \frac{u''(c(\bar{x}))}{u'(c(\bar{x}))}c'(\bar{x})(\bar{y} - \mathbb{E}[y'])$$

Using results from Carroll and Kimball (1996) in Lemmas 1 and 2, we can show $c'(x) > 0$ and $c''(x) < 0$, and therefore $c'(\bar{x}') < \infty$. Furthermore, $\bar{y} \geq \mathbb{E}[y']$. Since we assumed that $u(c)$ satisfies DARA, intuitively we would expect that risk aversion decreases as the agent's wealth increases. However, we need a stronger condition, that risk aversion is strictly decreasing in wealth. If this is satisfied then:

$$\lim_{x\to\infty} -\frac{u''(c(\bar{x}'))}{u'(c(\bar{x}'))} = 0$$

where $\bar{x}' = (1+r)a'(x) + \bar{y}$ is a function of x. Compare CRRA utility, which has strictly decreasing risk aversion, and CARA utility, which does not, to confirm the proposition. Therefore,

$$\frac{\mathbb{E}[u'(c(x))]}{u'(c(\bar{x}))} \approx 1$$

as desired. ■

Given these lemmas, for x sufficiently large:

$$u'(c(x_t)) = \beta(1+r)u'(c(\bar{x}_{t+1}))$$

$$u'(c(x_t)) < u'(c(\bar{x}_{t+1}))$$

$$c(x_t) > c(\bar{x}_{t+1})$$

$$x_t > \bar{x}_{t+1}$$

where the first line uses lemma 2, the second line uses $\beta(1+r) < 1$, third line uses $u''(c) < 0$, and the last line uses $c'(x) > 0$. Therefore, we know that savings does not diverge to infinity.

We can actually tighten the characterization for the transition function for cash-in-hand. We will now show there exists a x^* such that $\mathbb{E}_t[x_{t+1}(x^*)] = x^*$. We have already shown for x sufficiently large, $\mathbb{E}_t[x_{t+1}(x)] < x$ due to the continuity of $x_{t+1}(x)$ with respect to income shocks y_{t+1}.

Now we show that for x sufficiently small, $x_{t+1}(x) > x$. Show this by direct inspection of the transition for cash-in-hand. For all y_{t+1},

$$\begin{aligned}
\frac{x_{t+1}(x_t)}{x_t} &= \frac{(1+r)(x_t - c_t(x_t)) + y_{t+1}}{x_t} \\
&= (1+r)\left(1 - \frac{c_t(x_t)}{x_t}\right) + \frac{y_{t+1}}{x_t} \\
&\geq (1+r)\left(1 - \frac{c_t(x_t)}{x_t}\right) + \frac{\underline{y}}{x_t}
\end{aligned}$$

For simplicity assume that either savings is constrained $a_{t+1}(x) \geq 0$ with $a_{t+1}(0) = 0$, or follow Carroll (2012) to apply l'Hopital's Rule to show:

$$\lim_{x \to 0} \frac{c_t(x)}{x} = \lim_{x \to 0} \frac{c_t'(x)}{1} = \lim_{x \to 0} c_t'(x) < 1$$

In either case, we can show that:

$$\lim_{x_t \to 0} \frac{x_{t+1}(x_t)}{x_t} \geq (1+r)\left(1 - \lim_{x_t \to 0} \frac{c_t(x_t)}{x_t}\right) + \lim_{x_t \to 0} \frac{\underline{y}}{x_t} > \lim_{x_t \to 0} \frac{\underline{y}}{x_t}$$

Therefore, for x_t sufficiently small, $x_{t+1}(x_t) > x_t$.

By continuity of $x_{t+1}(x)/x$ with respect to x and with respect to y_{t+1}, we know that there exists a x^* that satisfies:

$$\mathbb{E}_t[x_{t+1}(x^*)] = x^* \tag{15}$$

Lastly, we will show that x^* is stable, in the sense that for $\epsilon > 0$, if $x \in (x^*, x^* + \epsilon)$

then $x_{t+1}(x) < x$. First, use the definition of x^* to characterize the policy function for consumption at x^*:

$$\begin{aligned}
\mathbb{E}_t[x_{t+1}(x^*)] &= (1+r)(x^* - c(x^*)) + 1 \\
x^* &= (1+r)(x^* - c(x^*)) + 1 \\
(1+r)c(x^*) &= 1 + rx^*
\end{aligned}$$

where in the first line, $\mathbb{E}_t[y_{t+1}] = 1$ by assumption. Now we will show that the expected change in $x_{t+1}(x)/x$ with respect to x is negative in a neighborhood of x^*.

$$\begin{aligned}
\mathbb{E}_t\left[\frac{\partial}{\partial x}\frac{x_{t+1}(x)}{x}\bigg| x = x^*\right] &= \mathbb{E}_t\left[\frac{\partial}{\partial x}(1+r)\left(1 - \frac{c_t(x)}{x}\right) + \frac{\partial}{\partial x}\frac{y_{t+1}}{x}\bigg| x = x^*\right] \\
&= \mathbb{E}_t\left[(1+r)\left(\frac{c_t(x) - c_t'(x)x}{x^2}\right) - \frac{y_{t+1}}{x^2}\bigg| x = x^*\right] \\
&= \frac{1}{(x^*)^2}\Big((1+r)\left(c_t(x^*) - c_t'(x^*)x^*\right) - \mathbb{E}_t\left[y_{t+1}|x = x^*\right]\Big) \\
&= \frac{1}{(x^*)^2}\Big((1+rx^*) - (1+r)c_t'(x^*)x^* - 1\Big) \\
&= \frac{1}{x^*}\Big((1+r)(1 - c_t'(x^*)) - 1\Big) \\
&< \frac{1}{x^*}\Big((1+r)\beta^{\frac{1}{\sigma}}(1+r)^{\frac{1}{\sigma}-1} - 1\Big) \\
&= \frac{1}{x^*}\Big(\underbrace{(\beta(1+r))^{\frac{1}{\sigma}}}_{<1} - 1\Big) \\
&< 0
\end{aligned}$$

where the second inequality follows from the assumption that $\beta(1+r) < 1$ and the first inequality follows from a result we will prove in the following section: when income is deterministic then $c_t'(x|y_t = \bar{y}) = 1 - \beta^{\frac{1}{\sigma}}(1+r)^{\frac{1}{\sigma}-1}$. This is important when combined with stochastic income and DARA preferences, since these imply that the consumption policy function is concave (Lemma 2 above). Then $c_t'(x) > c_t'(x|y_t = \bar{y})$, which means that the marginal propensity to consume is larger with stochastic income than with deterministic income.

In section 4.2, we will follow Huggett (1993) in generalizing this result to an autoregressive process for income (in particular a two state Markov process). See Chamberlain and Wilson (2000) for an extension to a more general stochastic process (any process that satisfies a regularity condition on the volatility of shocks). See Carroll (2012) for proofs of these results with an income process that contains a unit root.

3.3 Marginal Propensity to Consume

One of the insights in the last section 3.2 is that when income is uncertain, agents must be impatient $\beta(1+r) < 1$ in order for consumption and savings allocations to converge. As we just showed, the mechanism underlying the result hinges on the fact that, under these two conditions, agents have a larger marginal propensity to consume out of total wealth. We now explore the implications of the Buffer Stock model for the marginal propensity to consume[6].

Permanent and Transitory Shocks: We will consider a more general income process for this investigation. Suppose income Y can be decomposed into a permanent and transitory component such that:

$$\begin{aligned} Y_t &= \Gamma_t z_t \\ \Gamma_t &= \Gamma_{t-1} g_t \\ \log(g_{t+1}) &= (1-\rho_g)\log(1+\gamma) + \rho_g \log(g_t) + \eta_{t+1} \\ \log(z_{t+1}) &= (1-\rho_z)\mu + \rho_z \log(z_t) + \varepsilon_{t+1} \\ \eta &\sim \mathcal{N}(-\sigma_\eta^2/2, \sigma_\eta^2) \\ \varepsilon &\sim \mathcal{N}(-\sigma_\varepsilon^2/2, \sigma_\varepsilon^2) \end{aligned}$$

where we will assume that $\rho_g = \rho_z = \mu = 0$ and $\gamma > 0$. Then the stochastic process is:

$$\begin{aligned} Y_t &= \Gamma_t z_t \\ \Gamma_t &= \Gamma_{t-1} g_t \\ z_t &= \exp(\varepsilon_t) \end{aligned}$$

[6] This section closely follows Chris Carroll's lecture notes as well as Carroll (2009).

$$g_t = (1+\gamma)\exp(\eta_t)$$

First, this means that the income shock component z is purely transitory. Second, the permanent shock has a deterministic trend, $1+\gamma$, and transitory growth-rate shocks, η, that perturb income growth from the deterministic trend. To gain intuition, we can iterate backward to write the permanent component as:

$$\Gamma_t = (1+\gamma)^t \exp\left(\sum_{j=0}^{t} \eta_j\right)$$

The consumption-savings problem can now be rewritten as:

$$\begin{aligned} V_t(A_t, \Gamma_t, z_t, g_t) = \max_{C_t, A_{t+1}} \; & u(C_t) + \beta \mathbb{E}_t \left[V_{t+1}(A_{t+1}, \Gamma_{t+1}, z_{t+1}, g_{t+1}) \right] \\ \text{s.t.} \;\; C_t + A_{t+1} &\leq Y_t + (1+r)A_t \\ Y_t &= \Gamma_t z_t \\ \Gamma_{t+1} &= \Gamma_t g_{t+1} \end{aligned}$$

Let $u(c) = c^{1-\sigma}/(1-\sigma)$. Since ε and η are iid shocks, we can rewrite the problem with cash-in-hand as a state variable:

$$\begin{aligned} V_t(X_t, \Gamma_t) = \max_{C_t} \; & u(C_t) + \beta \mathbb{E}_t \left[V_{t+1}(X_{t+1}, \Gamma_{t+1}) \right] \\ \text{s.t.} \;\; X_{t+1} &= (1+r)(X_t - C_t) + Y_{t+1} \\ Y_{t+1} &= \Gamma_{t+1} z_{t+1} \\ \Gamma_{t+1} &= \Gamma_t g_{t+1} \end{aligned}$$

It will be convenient to define variables that have been normalized by Γ_t as lower case letters. For example, normalized income is: $y_t = z_t$. Because the utility function is homogeneous of degree $1-\sigma$ and the constraint set is linear, we can show that the value function is also homogeneous of degree $1-\sigma$. Then we can rewrite the problem in normalized form:

$$\begin{aligned} \Gamma_t^{1-\sigma} v(x) = \max_{c} \; & \Gamma_t^{1-\sigma} \frac{c^{1-\sigma}}{1-\sigma} + \beta \mathbb{E}_t \left[\Gamma_{t+1}^{1-\sigma} v(x') \right] \\ \text{s.t.} \;\; \Gamma_{t+1} x' &= (1+r)\Gamma_t (x - c) + \Gamma_{t+1} z' \end{aligned}$$

and dividing through by the permanent component of income:

$$v(x) = \max_c \frac{c^{1-\sigma}}{1-\sigma} + \beta(1+\gamma)^{1-\sigma}\mathbb{E}\left[e^{(1-\sigma)\eta'}v(x')\right] \tag{16}$$

$$\text{s.t.} \quad x' = e^{-\eta'}\frac{1+r}{1+\gamma}(x-c) + e^{\varepsilon'}$$

where $g = (1+\gamma)\exp(\eta)$ and $z = \exp(\varepsilon)$ with $\mathbb{E}[z] = 1$. We will work with this final formulation of the problem in equation (16). The associated Euler equation is:

$$c^{-\sigma} = \frac{\beta(1+r)}{(1+\gamma)^{\sigma}}\mathbb{E}\left[\left(e^{\eta'}c'\right)^{-\sigma}\right] \tag{17}$$

Consumption and savings will converge if the effective discount factor is bounded above by one (known as the "impatience condition"):

$$\beta(1+r)\mathbb{E}\left[\frac{1}{[(1+\gamma)e^{\eta'}]^{\sigma}}\right] < 1 \tag{18}$$

Deterministic Income: To start, it will be instructive to consider the case in which $z_t = 1$ and $\eta_t = 0$ for all t. Then:

$$Y_t = \Gamma_t = (1+\gamma)^t$$

Additionally, assume that $\gamma < r$.

Consider the unnormalized problem. The deterministic Euler equation in (17) becomes:

$$C_t^{-\sigma} = \beta(1+r)C_{t+1}^{-\sigma}$$

then we can write consumption as:

$$C_{t+j} = \left((\beta(1+r))^{\frac{1}{\sigma}}\right)^j C_t$$

Appealing to the intertemporal budget constraint in equation (7), we can iterate forward on the budget constraint and sum:

$$\sum_{j=0}^{\infty}\left(\frac{1}{1+r}\right)^j C_{t+j} = (1+r)A_t + \sum_{j=0}^{\infty}\left(\frac{1+\gamma}{1+r}\right)^j \Gamma_t$$

$$= (1+r)A_t + \frac{1+r}{r-\gamma}\Gamma_t$$

$$\equiv W_t + H_t$$

where $W_t \equiv (1+r)A_t$ is financial wealth and H_t corresponds to human wealth in equation (7). Now, we can use the Euler equation to rewrite the left hand side:

$$\sum_{j=0}^{\infty}\left(\frac{1}{1+r}\right)^j C_{t+j} = \sum_{j=0}^{\infty}\left(\frac{1}{1+r}\right)^j \left((\beta(1+r))^{\frac{1}{\sigma}}\right)^j C_t$$

$$= \left(\frac{1}{1-\beta^{\frac{1}{\sigma}}(1+r)^{\frac{1}{\sigma}-1}}\right) C_t$$

Therefore, the budget constraint gives:

$$C_t = \left(1 - \frac{(\beta(1+r))^{\frac{1}{\sigma}}}{1+r}\right)\left(W_t + H_t\right)$$

Notice that cash-in-hand is:

$$X_t = W_t + \Gamma_t \quad \Longrightarrow \quad x_t = w_t + 1$$

Then if we normalize each variable by the permanent income component Γ_t and substitute $w_t = x_t - 1$, then:

$$c(x_t) = \left(1 - \frac{(\beta(1+r))^{\frac{1}{\sigma}}}{1+r}\right)\left(x_t + \bar{h} - 1\right)$$

Now we can find the **marginal propensity of consumption out of transitory income**. In particular, how much more of a good will an agent consume if given an additional unit (or $\epsilon > 0$) of wealth $w_t = x_t - 1$:

$$c'(x_t) = \left(1 - \frac{(\beta(1+r))^{\frac{1}{\sigma}}}{1+r}\right)$$

To provide some content to this analytical expression, consider parameter values: Then

β	r	σ
0.96	0	2.0

$c'(x) = 1 - 0.96^{1/2} \approx 0.0202$, which is a very small consumption response to an increase in transitory income shocks.

Now let's find the **marginal propensity of consumption out of permanent income**:

$$\frac{\partial C_t}{\partial \Gamma_t} = \left(1 - \frac{(\beta(1+r))^{\frac{1}{\sigma}}}{1+r}\right)\frac{1+r}{r-\gamma}$$

where $H_t = \Gamma_t(1+r)/(r-\gamma)$ and $\partial W_t/\partial\Gamma_t = 0$.

Notice that if the "impatience condition" is violated, such that $\beta(1+r) = (1+\gamma)^\sigma$ then:

$$\frac{\partial C_t}{\partial \Gamma_t} = \left(1 - \frac{1+\gamma}{1+r}\right)\frac{1+r}{r-\gamma} = 1$$

but if it is not violated, $\beta(1+r) < (1+\gamma)^\sigma$ then:

$$\frac{\partial C_t}{\partial \Gamma_t} = \left(1 - \frac{(\beta(1+r))^{\frac{1}{\sigma}}}{1+r}\right)\frac{1+r}{r-\gamma} > \left(1 - \frac{1+\gamma}{1+r}\right)\frac{1+r}{r-\gamma} = 1$$

and the MPC out of permanent income is greater than one. Furthermore, when $(1+\gamma)^\sigma$ increases relative to $\beta(1+r)$, agents increase consumption. This reflects the effect of a higher growth trend: agents would like to smooth consumption (due to concave utility) over time, so if consumption is growing at a faster rate agents wish to consume more today. Generally, permanent shocks to income are shocks to the growth rate of income and affect today's consumption, in part, through agents' desire to smooth.

To build intuition, again let's set parameter values. Keeping (β, σ) from before now assume that $r = 0.02$ and $\gamma = 0.01$. First notice that the impatience condition is satisfied: $\beta(1+r) \approx 0.98 < 1.02 \approx (1+\gamma)^\sigma$. Then $\partial C_t/\partial\Gamma_t \approx 0.0202 \cdot 102 = 2.06$. As a comparative static, suppose $g = 0.015$ then the impatience condition still holds but now $\partial C_t/\partial\Gamma_t \approx 4.12$ which is larger as expected (and as a derivative with respect to γ readily informs us).

Stochastic Income: Now suppose again that $z_t = \exp(\varepsilon_t)$ such that $\varepsilon \overset{iid}{\sim} \mathcal{N}(-\sigma_\varepsilon^2/2, \sigma_\varepsilon^2)$, and that $\eta \overset{iid}{\sim} \mathcal{N}(-s_\eta^2/2, s_\eta^2)$.

The **marginal propensity to consume out of transitory income** takes a familiar form:

$$\frac{\partial c_t(x)}{\partial \varepsilon_t} = c_t'(x) \cdot \frac{\partial x}{\partial \varepsilon_t} = c_t'(x)$$

If we take the limit as the variance of ε_t goes to zero, then we will recover the deterministic marginal propensity to consume out of transitory shocks. However, when σ_ε^2 is strictly positive, we must rely on computation to obtain a characterization of $c_t'(x)$. The following table is taken from Carroll (1997) and gives numerical results from the model corresponding to (1) average MPC out of wealth (transitory shocks), (2) average net wealth and (3) target net wealth. The table reports the baseline parameterization and then reports results from recomputing with one parameter changed at a time. The baseline parameters are: The

Table 1: Baseline Parameters

γ	β	r	σ	σ_η^2	σ_ε^2
0.02	0.96	0	2.0	0.1	0.1

comparative statics are as follows: The table shows that, for example, when agents become

Table 2: Baseline and Parameter Deviations

	(1)	(2)	(3)
Baseline	0.33	0.35	0.32
$\gamma = 0$	0.16	0.66	0.62
$\gamma = 0.04$	0.42	0.28	0.25
$\beta = 1$	0.15	0.66	0.61
$\beta = 0.90$	0.46	0.25	0.23
$r = 0.02$	0.26	0.45	0.42
$r = 0.04$	0.17	0.65	0.61
$\sigma = 1$	0.49	0.14	0.11
$\sigma = 5$	0.14	1.13	1.08
$\sigma_\eta^2 = 0.05$	0.38	0.3	0.28
$\sigma_\eta^2 = 0.15$	0.22	0.51	0.47
$\sigma_\varepsilon^2 = 0.05$	0.33	0.32	0.3
$\sigma_\varepsilon^2 = 0.15$	0.32	0.39	0.35

more risk averse, the MPC out of transitory income shocks becomes much lower while net wealth (or savings) because much larger. Similarly, if transitory shocks increase in variance, then the MPC decreases while net wealth increases. These responses are expected from a model with a precautionary motive.

Now we turn to the **marginal propensity to consume out of permanent income**. In order for the elasticity of consumption with respect to permanent income Γ_t to be different from one in this environment, it must be the case that permanent income is correlated to cash-in-hand. Because of the structure of permanent income, there is in fact a correlation:

$$cov\left(\Gamma_t, x_t\right) = cov\left(\Gamma_{t-1}(1+\gamma)e^{\eta_t}, e^{-\eta_t}\frac{1+r}{1+\gamma}a_{t+1} + \varepsilon_t\right)$$

$$= \Gamma_{t-1} \frac{1+r}{1+\gamma} a_{t+1} \cdot cov\left(e^{\eta_t}, e^{-\eta_t}\right)$$

where ε and η are iid random variables and therefore $cov(\varepsilon, \eta) = 0$.

We will define the marginal propensity to consume out of a permanent income as the consumption to response to a shock to permanent income at $t+1$. We will compute it as:

$$(1+\gamma)\Gamma_t \mu(a_{t+1}) \equiv \mathbb{E}_t \left[\frac{\partial}{\partial e^{\eta_{t+1}}} (1+\gamma)\Gamma_t e^{\eta_{t+1}} c(x_{t+1}) \right]$$

$$= (1+\gamma)\Gamma_t \mathbb{E}_t \left[c(x_{t+1}) + e^{\eta_{t+1}} c'(x_{t+1}) \frac{\partial x_{t+1}}{\partial e^{\eta_{t+1}}} \right]$$

$$\mu(a_{t+1}) = \mathbb{E}_t \left[c(x_{t+1}) - c'(x_{t+1}) e^{-\eta_{t+1}} \frac{1+r}{1+\gamma} a_{t+1} \right]$$

This is the proportion of the change in C_{t+1} to the change in Γ_{t+1} caused by the shock to permanent income η_{t+1}. As we discussed in section 3.2.2 and is shown in Carroll and Kimball (1996), uncertainty increases incentives to save and therefore decreases consumption c_{t+1}. Furthermore, the precautionary savings motive increases the consumption response to a change in wealth $c'(x_{t+1})$ especially for low values of x_{t+1}.

Therefore, *precautionary savings* <u>*reduces*</u> *the marginal propensity to consume out of permanent income,* $\mu(a_{t+1})$. Furthermore, the decrease in the marginal propensity to consume out of permanent shocks is directly related to the increase in the marginal propensity to consume out of transitory shocks, $c'(x_{t+1})$.

Lastly, notice that as we take the limit of the variances of both shocks to zero, $\sigma^2_\varepsilon \to 0$ and $\sigma^2_\eta \to 0$, the MPC out of permanent income converges to the deterministic case:

$$\mu(a_{t+1}) = c(x_{t+1}) - c'(x_{t+1}) \frac{1+r}{1+\gamma} a_{t+1}$$

$$= \left(1 - \frac{(\beta(1+r))^{\frac{1}{\sigma}}}{1+r}\right) \left(w_{t+1} + \bar{h}\right) - \left(1 - \frac{(\beta(1+r))^{\frac{1}{\sigma}}}{1+r}\right) \frac{1+r}{1+\gamma} a_{t+1}$$

$$= \left(1 - \frac{(\beta(1+r))^{\frac{1}{\sigma}}}{1+r}\right) \left(\frac{1+r}{1+\gamma} a_{t+1} + \frac{1+r}{r-\gamma} - \frac{1+r}{1+\gamma} a_{t+1}\right)$$

$$= \left(1 - \frac{(\beta(1+r))^{\frac{1}{\sigma}}}{1+r}\right) \frac{1+r}{r-\gamma}$$

where we have already shown that (but now with $\gamma \neq 0$):

$$\bar{h} = H_{t+1}/\Gamma_{t+1} = \frac{1+r}{r-\gamma}$$

$$w_{t+1} = W_{t+1}/\Gamma_{t+1} = \frac{1+r}{1+\gamma}(x_t - c(x_t)) = \frac{1+r}{1+\gamma}a_{t+1}$$

Therefore, the deterministic income case corresponds to $\partial C_{t+1}/\partial \Gamma_{t+1}$ from before, as expected.

MPC and Savings: Recall the analysis following equation (15), in which we showed that there exists a x^* such that $\mathbb{E}_t[x_{t+1}(x^*)] = x^*$ and $x_{t+1}(x)$ is stable when x is in an ε-neighborhood of x^*. We will use those results to evaluate the marginal propensity to consume out of permanent shocks at the stable point. We will also characterize how the marginal propensity to consume out of permanent shocks covaries with savings.

We prove two propositions.

1. *Proposition 1:* If $a^* \equiv x^* - c(x^*) > 0$, then $\mu(a^*) < 1$.

   Proof: We will evaluate the definition $\mathbb{E}_t[a_{t+2}(a^*)] = a^*$:

$$\mathbb{E}_t[a_{t+2}] = \mathbb{E}_t\left[x_{t+1} - c(x_{t+1})\right]$$

$$a^* = \mathbb{E}_t\left[e^{-\eta_{t+1}}\frac{1+r}{1+\gamma}a^* + z_{t+1} - c\left(e^{-\eta_{t+1}}\frac{1+r}{1+\gamma}a^* + z_{t+1}\right)\right]$$

$$\mathbb{E}_t\left[c\left(e^{-\eta_{t+1}}\frac{1+r}{1+\gamma}a^* + z_{t+1}\right)\right] = 1 + \left(\frac{1+r}{1+\gamma}\mathbb{E}_t\left[e^{-\eta_{t+1}}\right] - 1\right)a^*$$

   where, as we assumed, $\mathbb{E}[z] = 1$. Now substitute this expression into the marginal propensity to consume out of permanent income:

$$\mu(a^*) = \mathbb{E}_t\left[c(x_{t+1}) - c'(x_{t+1})e^{-\eta_{t+1}}\frac{1+r}{1+\gamma}a_{t+1} \;\middle|\; a_{t+1} = a^*\right]$$

$$= 1 + \left(\frac{1+r}{1+\gamma}\mathbb{E}_t\left[e^{-\eta_{t+1}}\right] - 1\right)a^* - \mathbb{E}_t\left[c'(x_{t+1})e^{-\eta_{t+1}}\frac{1+r}{1+\gamma}a_{t+1} \;\middle|\; a_{t+1} = a^*\right]$$

$$< 1 + \left(\frac{1+r}{1+\gamma}\mathbb{E}_t\left[e^{-\eta_{t+1}}\right] - 1\right)a^* - \left(1 - \frac{\left(\beta(1+r)\right)^{\frac{1}{\sigma}}}{1+r}\right)\frac{1+r}{1+\gamma}\mathbb{E}_t\left[e^{-\eta_{t+1}}\right]a^*$$

$$= 1 + \left(\frac{\left(\beta(1+r)\right)^{\frac{1}{\sigma}}}{1+\gamma}\mathbb{E}_t\left[e^{-\eta_{t+1}}\right] - 1\right)a^*$$

where the inequality in line 3 is due to the precautionary motive: $c'(x)$ is larger under income uncertainty than when income is deterministic.

Since $\eta \sim \mathcal{N}(-\sigma_\eta^2/2, \sigma_\eta^2)$, we know that $\exp(\eta)$ is log-normally distributed with mean one. Then for $\kappa > 0$ we know $\mathbb{E}[\exp(\kappa\eta)] = \exp(\kappa\sigma_\eta^2/2 + \kappa^2\sigma_\eta^2/2)$. Then we can rewrite the impatience condition as:

$$\frac{\beta(1+r)}{(1+\gamma)^\sigma}e^{\frac{\sigma_\eta^2}{2}\sigma + \frac{\sigma_\eta^2}{2}\sigma^2} < 1 \tag{19}$$

But then:

$$\begin{aligned}
\frac{\left(\beta(1+r)\right)^{\frac{1}{\sigma}}}{1+\gamma}\mathbb{E}_t\left[e^{-\eta_{t+1}}\right] &= \frac{\left(\beta(1+r)\right)^{\frac{1}{\sigma}}}{1+\gamma}e^{(-1)\frac{-\sigma_\eta^2}{2} + (-1)^2\frac{\sigma_\eta^2}{2}} \\
&= \frac{\left(\beta(1+r)\right)^{\frac{1}{\sigma}}}{1+\gamma}e^{\sigma_\eta^2} \\
&< \frac{\left(\beta(1+r)\right)^{\frac{1}{\sigma}}}{1+\gamma}\left(e^{\sigma_\eta^2}\right)^{\frac{1}{2}(\sigma+\sigma^2)} \\
&= \left(\frac{\beta(1+r)}{(1+\gamma)^\sigma}e^{\frac{\sigma_\eta^2}{2} + \frac{\sigma_\eta^2}{2}\sigma}\right)^{\frac{1}{\sigma}} \\
&< 1^{\frac{1}{\sigma}} = 1
\end{aligned}$$

where the inequality in line 3 holds because $\sigma > 1$.

But if $a^* > 0$ then:

$$\mu(a^*) = 1 + \Bigg(\underbrace{\frac{\left(\beta(1+r)\right)^{\frac{1}{\sigma}}}{1+\gamma}\mathbb{E}_t\left[e^{-\eta_{t+1}}\right] - 1}_{<0}\Bigg)a^* < 1$$

Which proves that the marginal propensity to consume out of permanent income shocks is less than one at the savings target. ■

2. *Proposition 2:* Given a_{t+1}, the marginal propensity to consume out of permanent shocks is increasing in the level of savings:

$$\frac{\partial}{\partial a_{t+1}}\mu(a_{t+1}) > 0$$

Proof: Take a straightforward derivative:

$$\begin{aligned}
\frac{\partial \mu(a_{t+1})}{\partial a_{t+1}} &= \frac{\partial}{\partial a_{t+1}}\mathbb{E}_t\left[c(x_{t+1}) + c'(x_{t+1})e^{-\eta_{t+1}}\frac{1+r}{1+\gamma}a_{t+1}\right] \\
&= \mathbb{E}_t\left[c'(x_{t+1})e^{-\eta_{t+1}}\frac{1+r}{1+\gamma} - c''(x_{t+1})\left(e^{-\eta_{t+1}}\frac{1+r}{1+\gamma}\right)^2 a_{t+1} - c'(x_{t+1})e^{-\eta_{t+1}}\frac{1+r}{1+\gamma}\right] \\
&= \mathbb{E}_t\left[-c''(x_{t+1})\left(e^{-\eta_{t+1}}\frac{1+r}{1+\gamma}\right)^2 a_{t+1}\right] \\
&> 0
\end{aligned}$$

In Lemma 2 of section 3.2.2 we showed that the consumption function is concave, so $c''(x) < 0$. Therefore, given that $a_{t+1} = x_t - c(x_t) \geq 0$, the marginal propensity to consume out of permanent shocks is increasing with savings. ■

Computational Experiments: Following Carroll (2009), we compute the agent's policy function and simulate individual behavior for 20,000 agents over a large number of periods. We then compute statistics from the resulting stationary distribution. The details of such computation will be explained in Section 4 of these notes.

The model has parameters $\{\sigma, \beta, r, \gamma, \sigma_\eta, \sigma_\varepsilon\}$ that we must map to the data. We set $\sigma = 3$, which is standard in the literature. We choose $r = 0.03$ to correspond to the post-war average return on corporate bonds. We choose the standard value of $\beta = 0.96$, which is chosen to match a post-war average capital-output ratio of approximately 3 in representative agent models.

Following Carroll (1992), one could use the Panel Study of Income Dynamics (PSID) to recover the variances of permanent and transitory shocks from household-level income data: $\sigma_\eta = \sigma_\varepsilon = 0.10$ and $\gamma = 0.02$. Lastly, suppose that each period there is an iid shock of drawing zero income with a small probability $\phi = 0.005$. Below are the parameters obtained from the data.

Table 3: Baseline Parameters

γ	β	r	σ	σ_η	σ_ε^2
0.03	0.96	0.04	3.0	0.1	0.1

For explication, consider the following simplification. Set $r = 0$ so that β indexes the tightness of the impatience condition. As we discussed above, the tighter the impatience condition the closer to one the marginal propensity to consume out of permanent income is. The looser the condition, the larger the marginal propensity to consume out of permanent income (which is greater than one). Recall from equation (19):

$$\beta(1+r)\mathbb{E}_t\left[\left((1+\gamma)e^{\eta_{t+1}}\right)^{-\sigma}\right] = \beta(1+r)(1+\gamma)^{-\sigma}\exp\left(\sigma\frac{\sigma_\eta^2}{2}+\sigma^2\frac{\sigma_\eta^2}{2}\right)$$

$$\approx \beta(1+r)\cdot\left(\frac{1}{1+\sigma\gamma}\right)\cdot\left(1+\sigma\frac{\sigma_\eta^2}{2}+\sigma^2\frac{\sigma_\eta^2}{2}\right)$$

If $\gamma = 0.02$, and $\sigma_\eta^2 = 0.01$ then the impatience condition becomes:

$$\beta(1+r)\mathbb{E}_t\left[\left((1+\gamma)e^{\eta_{t+1}}\right)^{-\sigma}\right] \approx 0.96\cdot\frac{1+0.01\cdot\frac{1}{2}(3+9)}{1+3\cdot(0.02)} = 0.96 = \beta$$

To summarize, set parameters as follows:

Table 4: Example Parameters

γ	β	r	σ	σ_η^2	σ_ε^2
0.02	0.96	0.0	3.0	0.01	0.1

Therefore, closely following Chris Carroll's lecture notes, our experiment is to vary β across a range of values and measure the average savings a, average marginal propensity to consume out of transitory shocks $c'(x)$, and the average marginal propensity to consume out of permanent income $\mu(a)$. The results are in table 5 below. A key result of table 5 is that the marginal propensity to consume out of permanent income shocks is fairly inelastic with respect to changes in β. From the definition of $\mu(a^*)$, we know that changes in β affect the target savings level a^*, and in turn has an impact on both consumption $c(x)$ and the marginal propensity to consume out of wealth $c'(x)$. An increase in savings a^* decreases both the level of consumption and the marginal propensity to consume, which has a conflicting

Table 5: Comparative Statics with respect to β

β	Mean a_{t+1}	Mean $c'(x_t)$	Mean $\mu(a_{t+1})$
1.00	1.14	0.08	0.91
0.98	0.77	0.18	0.87
0.96	0.64	0.24	0.85
0.94	0.59	0.28	0.84

effect on $\mu(a^*)$. The computational experiment informs us that net effect of a change is small.

3.4 Empirical Evaluation

Ludvigson and Michaelides (2001) evaluate whether buffer-stock saving can explain the excess smoothness and excess sensitivity of consumption puzzles.

Excess Sensitivity: Recall that excess sensitivity refers to the empirical observation that consumption growth is significantly predicted by lagged income growth, while the Permanent Income Hypothesis predicts that consumption is random walk (e.g. orthogonal to any lagged variables). In fact, Ludvigson and Michaelides (2001) to estimate the relationship:

$$\Delta \log(C_t) = \beta_0 + \beta_1 \Delta \log(Y_{t-1}) + \epsilon_t$$

and find that β_1 is statistically different from zero and ranges between 0.14 and 0.18 depending on consumption measure and annual versus quarterly frequency data.

Excess Smoothness: The excess smoothness puzzle refers to the empirical observation that consumption growth has a lower standard deviation than income growth, while the Permanent Income Hypothesis predicts that consumption growth and income growth are equally volatile (e.g. both follow a random walk). In fact, Ludvigson and Michaelides (2001) find that, in US annual data, the standard deviation of nondurable consumption growth is less than two-thirds (61%) as large as the standard deviation of annual wage and salary income growth. The authors also estimate the relationship:

$$\Delta y_t = (1-\rho)\mu + \rho \Delta y_{t-1} + \epsilon_t$$

and recover $\rho = 0.22$. According to the Permanent Income Hypothesis with a serially correlated AR(1) income process we know that:

$$\sigma_{\Delta c} = \frac{1}{1-\rho}\sigma_{\Delta y}$$

so that the model predicts that consumption growth's standard deviation is $1/(1-.22) = 1.26$ times larger than income growth's standard deviation.

Buffer-Stock Model: Ludvigson and Michaelides (2001) simulate panel data from the buffer-stock savings model and then re-estimate the above relationships for the simulated data. If the relationships from the simulated data reasonably match the relationships we observe in the data, then one may conclude that the buffer-stock model is successful at resolving the excess sensitivity and excess smoothness puzzles.

The authors consider two versions of the buffer-stock model: one in which the income process is estimated from annual data and another in which the income process is estimated from quarterly data. It turns out that the annual income growth process is best represented as a random walk, while the quarterly income growth process is best represented as an AR1 with persistence $\rho = 0.26$.

Ludvigson and Michaelides (2001) find that the model with an annual random walk income growth process generates $\sigma_{\Delta c}/\sigma_{\Delta y} = 0.99$, which is almost exactly what the Permanent Income Hypothesis would predict ($\sigma_{\Delta c}/\sigma_{\Delta y} = 1$). This result may seem at odds with the prediction from table 5 that the marginal propensity to consume out of permanent income shocks (e.g. growth shocks) is less than one. However, $\mu(a)$ measures the *immediate response* of consumption to permanent income shocks, while the data reflects the total response over time. Therefore, the model and data are not necessarily inconsistent.

The model with a quarterly AR1 income growth process generates $\sigma_{\Delta c}/\sigma_{\Delta y} = 1.09$ versus a 1.26 in the "strict" Permanent Income Hypothesis model. The reason that the buffer-stock model generates more smooth consumption is that the buffer-stock consumer may not have sufficient wealth to increase consumption in response to shocks at the rate predicted by the Permanent Income Hypothesis (e.g. 1.26 times the size of the income shock). As a result, we see that consumption is smoother. Relative to the empirically observed $\sigma_{\Delta c}/\sigma_{\Delta y} = 0.68$, the buffer-stock model explains $(1.26 - 1.09)/(1.26 - 0.68) \approx 0.29$ of the gap.

Turning to the excess sensitivity puzzle, we expect the buffer stock to generate some consumption response to income changes. The buffer-stock model predicts that when an individual receives news about future income, if their wealth is low, an individual may delay their consumption response until receiving additional income. Therefore, the theory should generate sensitivity to income shocks, especially for low-wealth agents.

Ludvigson and Michaelides (2001) find that when income shocks follow a random walk, the model generates a very small consumption response to income changes. However, random walk income shocks are easily insured away through precautionary savings. When income shocks follow an AR1 process, the consumption growth response to lagged income growth is 0.06 which is notably larger than the Permanent Income Hypothesis' prediction of zero but still half of the empirically observed response of 0.16.

Incomplete Information: Ludvigson and Michaelides (2001) extend the baseline buffer-stock model by jointly including idiosyncratic permanent and transitory income shocks, and aggregate income shocks. The authors assume that agents can observe income each period, but cannot observe the magnitude of each shock. As a result, agents attribute some fraction of realized aggregate shocks to idiosyncratic transitory movements in income, as idiosyncratic income shocks have the highest probability of occurring. When agents perceive transitory shocks with higher frequency, consumption will become smoother relative to income volatility. However, if what was perceived as a positive idiosyncratic transitory shock was actually a positive aggregate income shock, then next period's income will be higher than expected and individuals will increase consumption in response to the innovation in income. Therefore, the incomplete-information buffer-stock model explains why there is a correlation between consumption growth and lagged income, and explains why consumption growth is smoother than income growth.

Model simulations show that the incomplete-information buffer-stock matches the data better than the complete-information version. For the annual model with random walk income shocks, the improvement is slight. However, the quarterly model with AR1 income process achieves $\sigma_{\Delta c}/\sigma_{\Delta y} = 0.91$ and a response of consumption growth to lagged income growth of 0.433. Compared to quarterly data, the incomplete-information model improves along the dimension of the excess smoothness puzzle and actually overshoots the empirically observed sensitivity to lagged income growth of 0.16.

4 Neoclassical Growth Model with Incomplete Markets

In this section we use the insights we developed in Section 3 and embed the consumption-savings problem into a general equilibrium framework. The individual consumer's problem provides a micro-foundation from which we can study aggregate variables and the aggregate implications of policies that affect individuals at the microeconomic level. In this section we will study stationary equilibria, equilibria in which aggregate variables are time invariant. In the next section we will study equilibria in which aggregate variables are time-varying.

4.1 Recursive Competitive Equilibrium

We will begin by describing the economic environment. The consumer side will look familiar, as it is the same basic model of Section 3. Following Aiyagari (1994), we will add a production sector, much like the representative firm of Section 1.4, that demands consumers' savings and labor as inputs to production. Lastly we will require that prices are endogenously determined by the interaction of consumers and the representative firm in factor markets. In particular, equilibrium in the asset market will endogenously yield an interest rate r such that $\beta(1+r) < 1$ due to precautionary motives. Note that under complete asset markets, the stationary equilibrium yields $\beta(1+r) = 1$ because agents can fully insure consumption.

Demographics: The economy is populated with a unit continuum of infinitely lived agents. These agents are ex-ante identical but will be ex-post heterogeneous along two dimensions: wealth and income.

Preferences: Each agent values streams of consumption $\{c_t\}_{t=0}^{\infty}$ according to:

$$U(\{c_t\}_{t=0}^{\infty}) = \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t u(c_t)$$

where $\beta < 1$ is the agent's discount factor and $u : \mathbb{R}_+ \to \mathbb{R}$ is the agent's instantaneous utility function satisfying $u' > 0$, $u'' < 0$, twice continuously differentiability and Inada conditions.

Endowments: Each agent is endowed with a unit of labor effort that is inelastically supplied. Agents' labor productivity is idiosyncratic and stochastic, denoted ε_t. Assume that labor productivity follows an AR1 process, where innovations to the process are iid across individuals and time.

For computational tractability and tighter characterization, we assume that labor productivity takes on a discrete and finite number of values $\varepsilon_t \in E \equiv \{\underline{\varepsilon}, \ldots, \overline{\varepsilon}\}$, and follows a Markov process with transition probabilities $\pi(\varepsilon'|\varepsilon)$. Assume that a law of large numbers holds (c.f. Judd (1985) and Uhlig (1996)) so that $\pi(\varepsilon'|\varepsilon)$ is also the fraction of agents that receive shock ε' tomorrow conditional on having received shock ε today.

Assume that the Markov transition function satisfies Perron-Frobenius conditions and therefore admits a stationary distribution over labor productivity, denoted $\pi^*(\varepsilon)$. Using the stationary distribution we can compute aggregate effective labor supply as:

$$\sum_{\varepsilon \in E} \pi^*(\varepsilon) \cdot \varepsilon$$

for all t.

Budget Set: The agent's budget constraint is given by:

$$c_t + a_{t+1} \leq w_t \varepsilon_t + (1 + r_t) a_t$$

where consumption c_t and savings a_{t+1} is afforded by labor income $w_t \varepsilon_t$ and wealth $(1+r_t)a_t$. Labor income is given by the wage w_t for each unit of labor productivity ε_t. Wealth is held in risk-free, one-period non-contingent bonds that yield interest r_{t+1} next period. Agents may also use bonds to borrow against future income if $a_{t+1} < 0$. However, agents face a borrowing constraint of the form:

$$a_{t+1} \geq \underline{a}_t$$

where $\underline{a}_t \in \mathbb{R}$ for all t. In what follows we will assume a *no borrowing constraint*, which sets $\underline{a}_t = 0$ for all t.

Production: Assume there is a representative firm that is endowed with a constant returns to technology production technology $Y_t = F(K_t, L_t)$ that satisfies Inada conditions and decreasing marginal returns in both capital K_t and labor L_t. The firm sells its output in perfectly competitive goods markets and purchases inputs in perfectly competitive factor markets, at prices w_t for labor and $r + \delta$ for capital. Physical capital depreciates at rate $\delta < 1$.

Distributions:[7] In an given period, an agent's individual state is his wealth and income shock, (a, ε). There is also an aggregate state of the economy, which is the joint distribution over wealth and income shocks, denoted $\lambda(a, \varepsilon)$. Using the distribution, we can aggregate over individual decisions and compute aggregate variables and prices.

Define $\bar{a}$ as the largest quantity of assets any agent would possibly hold. We will show that such an upper bound exists in Section 4.2, but for now define the compact set of asset holdings $A \equiv [\underline{a}, \bar{a}]$. Then we can define the agent's state space S as the Cartesian product $A \times E$. Let the Borel σ-algebra $\mathscr{B}$ be defined as the Kronecker product $B_A \otimes 2^E$, where B_A is the Borel σ-algebra on A and 2^E is the power set of E. Let $\mathcal{S} = (\mathcal{A} \times \mathcal{E})$ be a typical subset of $\mathscr{B}$. The space $(S, \mathscr{B})$ is a measurable space such that $\lambda(\mathcal{S})$ is the measure of agents in the set $\mathcal{S}$ for any set $\mathcal{S} \in \mathscr{B}$. Let $\Lambda(S, \mathscr{B})$ be the set of probability measures on $(S, \mathscr{B})$.

In order to construct a distribution, we must understand how individuals transition from state $(a, \varepsilon) \in S$ to state $(a', \varepsilon') \in S$ over time. Let $P : S \times \mathscr{B} \to [0, 1]$ be the transition function on $(S, \mathscr{B})$ defined by:

$$P\left((a, \varepsilon), \mathcal{A} \times \mathcal{E}\right) = \mathbb{1}\left[a'(a, \varepsilon) \in \mathcal{A}\right] \sum_{\varepsilon' \in \mathcal{E}} \pi(\varepsilon'|\varepsilon)$$

where $a'(a, \varepsilon)$ is the agent's optimal savings decision rule given state (a, ε) and $\mathbb{1}[\cdot]$ is the indicator function that equals one if the condition specified in brackets is satisfied and zero otherwise. The transition function $P((a, \varepsilon), \mathcal{A} \times \mathcal{E})$ specifies the probability that an agent with current state (a, ε) transitions to the set $\mathcal{A} \times \mathcal{E}$.

Given this transition function we can obtain a distribution over states that are realized next period from the distribution over states this period. Define $T^* : \Lambda(S, \mathscr{B}) \to \Lambda(S, \mathscr{B})$ as an operator that maps distributions into distributions according to:

$$(T^*\lambda)(\mathcal{A} \times \mathcal{E}) = \int_{A \times E} P\left((a, \varepsilon), \mathcal{A} \times \mathcal{E}\right) d\lambda \qquad \forall\ \mathcal{A} \times \mathcal{E} \in \mathscr{B}$$

Market Clearing: Denoting aggregate variables by capital letters, let C be aggregate consumption and Y be aggregate output. The aggregate resource constraint ensures that all consumption and investment is afforded by output:

$$C_t + K_{t+1} = Y_t + (1 - \delta)K_t$$

[7] For further background on measure theory, see Stokey, Lucas and Prescott (1989) chapters 7, 8.1 and 12.4.

Furthermore, asset markets must clear (savings equals investment):

$$K = \int_{A \times E} a'(a, \varepsilon) d\lambda$$

Lastly, labor markets must clear (labor demanded by firms equals labor exogenously supplied by individuals):

$$L = \sum_{\varepsilon \in E} \pi^*(\varepsilon) \cdot \varepsilon \quad \forall\, t$$

Stationary Recursive Competitive Equilibrium: We can finally define an equilibrium of this environment.

A *stationary recursive competitive equilibrium* is:

(i) policy functions for individual consumption $c : S \to \mathbb{R}$ and savings $a' : S \to \mathbb{R}$, and the associated value function $v : S \to \mathbb{R}$

(ii) optimal capital and labor for the representative firm, $(K, L) \in \mathbb{R}^2_+$,

(iii) prices $(r, w) \in \mathbb{R}^2_{++}$, and

(iv) a stationary probability measure $\lambda \in \Lambda(S, \mathscr{B})$

such that:

- Given prices (r, w), the policy functions (c, a') solve the individual's dynamic program with associated value function v:

$$v(a, \varepsilon) = \max_{c, a'} u(c) + \beta \sum_{\varepsilon' \in E} \pi(\varepsilon' | \varepsilon) v(a', \varepsilon')$$

$$\text{s.t.} \quad c + a' \leq w\varepsilon + (1 + r)a$$

$$a' \geq \underline{a}$$

- Given prices (r, w), the firm chooses K and L optimally such that $w = F_L(K, L)$ and $r = F_K(K, L) - \delta$,

- Given the distribution λ, the labor market clears:

$$L = \int_{A \times E} \varepsilon d\lambda = \sum_{\varepsilon \in E} \pi^*(\varepsilon) \cdot \varepsilon$$

the asset market clears:

$$K = \int_{A\times E} a'(a,\varepsilon)d\lambda$$

and the goods market clears (which trivially holds by Walras' Law):

$$\int_{A\times E} c(a,\varepsilon)d\lambda = F(K,L) - \delta K$$

- The distribution λ is a fixed point of the T^*-operator such that $\lambda = T^*(\lambda)$ or:

$$\lambda(\mathcal{A}\times\mathcal{E}) = \int_{A\times E} P\left((a,\varepsilon),\mathcal{A}\times\mathcal{E}\right)d\lambda$$

  for all $(\mathcal{A}\times\mathcal{E}) \in \mathscr{B}$, where the transition function P is constructed from $a'(a,\varepsilon)$ and $\pi(\varepsilon'|\varepsilon)$.

Note that in the individual's dynamic program, λ is also a state variable as it is needed to compute market clearing prices. However, because we have imposed stationarity on the aggregate allocation, the distribution λ is time invariant. Because λ does not change over time, it is not necessary to track its evolution and therefore does not need to be included explicitly as a state variable. Similarly, from the labor market clearing condition $L_t = L$ for all $t \geq 0$ because π^* and E are time invariant. And lastly, $K_t = K$ for all $t \geq 0$ since λ is time invariant.

4.2 Existence and Uniqueness

This section follows Huggett (1993) closely in proving that there exists a unique stationary distribution over wealth and income. Furthermore, these results extend the proofs of convergence for iid income shocks with $\beta(1+r) < 1$ in section 3.2.2. We then discuss Aiyagari's (1994) proof of existence of a market clearing asset price. Together, these proofs ensure us that we can construct an equilibrium of this economy numerical. We will visit computation in the next section.

Strategy: We construct an excess demand function for asset markets, show that the excess demand function is continuous and that it takes on both positive and negative values. Given these conditions, we can show that there exists a price r such that the excess demand function equals zero. We focus on the asset market because labor market clearing holds by construction (e.g. due to exogenous labor supply and the aggregate production technology) and because if asset markets clear then goods markets are guaranteed to clear by Walras' Law.

In the first part of this section we show that the excess demand function is continuous, in particular by showing that aggregate savings is continuous in the price r. Define asset supply (aggregate savings) as the function:

$$A(r) \equiv \int_{A\times E} a'(a,\varepsilon;r)\lambda^*(da,d\varepsilon;r)$$

Here we sketch the desired properties of the transition function Q and associated operator T^* that guarantee existence of a unique invariant distribution $\lambda(a,\varepsilon)$ for a given r:

- *Compactness:* We previously showed that there exists an upper bound on the asset space, $\bar{a}$ when $\beta(1+r) < 1$ and utility is of the DARA class. Given that income shocks are assumed to follow a discrete Markov process, the state space is compact.

- *Feller Property of P:* A transition function P on $(S,\mathscr{B})$ has the Feller Property if the operator T^* maps bounded and continuous functions into the space of bounded continuous function. Notice that the optimal policy $a'(a,\varepsilon)$ will be a mapping from $S \to S$ if the asset state space is compact and since E is assumed to be a finite, discrete set.

- *Monotonicity of P:* A transition function P on $(S,\mathscr{B})$ is monotone if for any bounded, continuous and increasing function $f : S \to \mathbb{R}$ the operator T^* generates a bounded, continuous and increasing function $T^*(f)$. We can show that the optimal policy function $a'(a,\varepsilon)$ is increasing in both arguments. Furthermore we can assume that the Markov process is increasing, so that $\pi(\bar{\varepsilon}|\bar{\varepsilon}) \geq \pi(\bar{\varepsilon}|\underline{\varepsilon})$ and $\pi(\underline{\varepsilon}|\underline{\varepsilon}) \geq \pi(\underline{\varepsilon}|\bar{\varepsilon})$. Then for any $f(a,\varepsilon)$ that is increasing in both arguments, $T^*(f)(a,\varepsilon)$ will place higher probability on a future event $(a',\varepsilon') > (a,\varepsilon)$ when we increase (a,ε).

- *Monotone Mixing Condition (MMC):* Roughly speaking, the monotone mixing condition requires that an agent starting from state $(a,\varepsilon) \in S$ will reach any arbitrary state $(a',\varepsilon') \in S$ within a finite time. This is sometimes referred to as the "American Dream, American Nightmare" condition because it implies full mobility across states over time.

  The condition generates stationarity since, if an agent were to hypothetically start from the state $(\bar{a},\bar{\varepsilon})$ and draw $\underline{\varepsilon}$ for T periods, where T is a large number, then because ε is governed by a mean-reverting process then the agent will spend his wealth until reaching a neighborhood of $\underline{a}$. The agent knows that these shocks are below average and his permanent income is higher than the draw $\underline{\varepsilon}$. Similar logic dictates that a consumer will save a fraction of income when he starts from $(\underline{a}, \underline{\varepsilon})$ and receives T periods of $\bar{\varepsilon}$ income draws.

Preliminaries: The economy in Huggett (1993) is slightly different than the production economy in the previous section. In particular Huggett's is a pure exchange economy in which agents sell debt and purchase borrowed funds at a price q. This price must clear markets: for every individual who sells a bond there must be an agent who purchases one. The bond price corresponds to the the interest rate: $q = 1/(1+r)$. Individuals do not supply labor, but instead collect a stochastic endowment that satisfies Markov transition matrix $\pi(\varepsilon'|\varepsilon)$. These differences are small and all proofs below transfer to the production economy with just a few cosmetic alterations.

First we state a preliminary theorem.

Theorem 1: Let $S = [\underline{a}, \infty) \times \{\underline{\varepsilon}, \bar{\varepsilon}\}$ and $C(S)$ the set of continuous, bounded functions on S. For $q > 0$ and $\underline{\varepsilon} + \underline{a} - q\underline{a} > 0$, there exists a unique solution, $v(a, \varepsilon; q) \in C(S)$, to

$$(Tv)(a, \varepsilon; q) = \max_{(c,a') \in \Gamma(a,\varepsilon;q)} \; u(c) + \beta \sum_{\varepsilon'} \pi(\varepsilon'|\varepsilon) v(a', \varepsilon'; q)$$

$$\Gamma(a, \varepsilon; q) = \left\{ (c, a') \;\middle|\; c + qa' \leq \varepsilon + a, \;\; a' \geq \underline{a} \right\}$$

and $T^n v_0$ converges uniformly to v as $n \to \infty$ from any $v_0 \in C(S)$.

Lemma: The solution $v(a, \varepsilon; q)$ is strictly increasing, strictly concave and continuously differentiable in a. There exist decision rules $\{c(a, \varepsilon; q), a'(a, \varepsilon; q)\}$ such that $a'(a, \varepsilon; q)$ is non-decreasing in a and strictly increasing in a for (a, ε, q) satisfying $a'(a, \varepsilon; q) > \underline{a}$.

Heuristic Proof: The proof is standard, except showing that Tv is a mapping from $C(S)$ into $C(S)$. The difficultly comes from the fact that utility is not bounded below (e.g. CRRA preferences). The gist of the proof is to show that c is bounded below and bounded away from zero because of Inada conditions. ■

Let $S = [\underline{a}, \bar{a}] \times \{\underline{\varepsilon}, \bar{\varepsilon}\}$ with Borel σ-algebra $\mathscr{B}$. Let $M(S)$ be the space of probability measures on $(S, \mathscr{B})$ defined by

$$(T^*\lambda)(B) = \int_S P(s, B) d\lambda \qquad \forall \; B \in \mathscr{B}$$

Theorem 2: If $q > 0$, $\underline{\varepsilon} + \underline{a} - q\underline{a} > 0$, $\beta < q$ and $\pi(\bar{\varepsilon}|\bar{\varepsilon}) \geq \pi(\bar{\varepsilon}|\underline{\varepsilon})$ then there exists a unique stationary probability measure λ (given q) on $(S, \mathscr{B})$, and $T^{*n}\lambda_0$ converges weakly to λ for any $\lambda_0 \in M(S)$.

Importance: The operator $T^*(\lambda)$ provides a method for checking whether $\{a'(a, \varepsilon; q), \lambda(a, \varepsilon; q), q\}$ is an equilibrium. Because λ_n converges weakly to the unique stationary probability measure $\lambda(q)$, and because $a'(a, \varepsilon; q)$ is a continuous and bounded function in $C(S)$, the following sequence of integrals converges:

$$\int_S a'(a, \varepsilon; q) d\lambda_n \to \int_S a'(a, \varepsilon; q) d\lambda(q)$$

Therefore, the limit of the operator gives aggregate savings which can be used to check whether market clearing holds for a particular q.

Heuristic Proof: The proof proceeds in two steps. In the first step, establish a compact set for savings so that $S = [\underline{a}, \bar{a}] \times \{\underline{\varepsilon}, \bar{\varepsilon}\}$. This is the ergodic set. This is done in three substeps:

- Show $a'(a, \underline{\varepsilon}) < a$ for all $a > \underline{a}$
- If $v_a(a, \varepsilon) > q^{-1}\beta \mathbb{E} v_a(a, \varepsilon')$ for all $a \geq a^* > \underline{a}$, then $a'(a, \varepsilon) < a$ for all $a \geq a^*$
- There exists a such that $a'(a, \bar{\varepsilon}) = a$

The second step establishes the unique existence of λ such that $T^{*n}\lambda_0 \to \lambda$. This is done in two substeps:

- For transition function P, if $\forall\ x, x' \in S$ s.t. $x \geq x'$, then

  $$\int_S \mathbf{1}\left[\tilde{x} \in B\right] P(x, d\tilde{x}) \geq \int_S \mathbf{1}\left[\tilde{x} \in B\right] P(x', d\tilde{x})$$

  where

  $$B \equiv \left\{ y \in S \;\middle|\; y \geq x, \text{ for some } x \in B \right\} \in \mathscr{B}_S$$

- Mixing holds: $\exists\ s^* \in S$, $\epsilon > 0$ and N such that

  $$P^N(\bar{a}, \{s \mid s \leq s^*\}) > \epsilon \qquad \text{and} \qquad P^N(\underline{a}, \{s \mid s \geq s^*\}) > \epsilon$$

Proof: Some preliminaries first: construct a transition function. Let $(S, \mathscr{B})$ be a measurable space with Borel σ-algebra on S. Let z be a random variable defined on the probability space $(Z, \mathscr{Z}, \lambda)$. Let $g : S \times Z \to S$ be a measurable function such that $s' = g(s, z)$. This structure induces a mapping $P : S \times \mathscr{B} \to [0, 1]$ defined by:

$$P(s, B) = \lambda\left(\{z \mid g(s, z) \in B\}\right) \quad \forall\, B \in \mathscr{B}$$

Hopenhayn and Prescott (1987), Lemma 5, provides conditions under which P is a transition function for a Markov process: if g is measurable in $S \times Z$ under the product σ-algebra, then P is a transition function for a Markov Process. The proof is simply that g is measurable with respect to (a, ε) by construction.

First Step (Compactness of State Space): Establish a compact set for savings so that $S = [\underline{a}, \bar{a}] \times \{\underline{\varepsilon}, \bar{\varepsilon}\}$. This is done in the following five lemmas.

Lemma 0 (Monotone Optimal Policies): If π is monotone, $u(\cdot)$ is strictly increasing and concave, and $u(a + \varepsilon - qa')$ exhibits increasing differences in (a', ε), then the policy function $a'(a, \varepsilon)$ is increasing in ε.

Proof: This is a straightforward application of Monotone Comparative Statics. Heuristically, first show that the utility over consumption in fact satisfies increasing differences in (a', ε). Then show that by induction that if $v(a, \varepsilon)$ satisfies first differences then so does $(Tv)(a, \varepsilon)$. This conclusion follows from the monotonicity of the transition function, $\pi(\varepsilon'|\varepsilon)$. Given these two results, an application of Theorem 10.6 in Sundaram (1996) provides the result. ■

Lemma 1: If $q > 0$, $\underline{\varepsilon} + \underline{a} - q\underline{a} > 0$, $\beta < q$ and $\pi(\bar{\varepsilon}|\bar{\varepsilon}) \geq \pi(\bar{\varepsilon}|\underline{\varepsilon})$, then $v'(a, \bar{\varepsilon}) \leq v'(a, \underline{\varepsilon})$ for $a'(a, \varepsilon) > \underline{a}$.

Proof: We will show that $v'(a, \bar{\varepsilon}) \leq v'(a, \underline{\varepsilon})$ for $a > \underline{a}$. By strict concavity of v, the result will follow.

First, define a sequence $\{v_n(a, \varepsilon)\}_{n=0}^{\infty}$ where $v_0(a, \varepsilon) = 0$ and $v_{n+1}(a, \varepsilon) = Tv_n(a, \varepsilon)$. Define the sequence of functions $\{a_{n+1}\}_{n=0}^{\infty}$ as the decision rules corresponding to each Tv_n. Proceed by induction to show $v_n'(a, \bar{\varepsilon}) \leq v_n'(a, \underline{\varepsilon})$ for each n. The base case $n = 0$ clearly holds: $0 = 0$. Now suppose the relationship holds at n and show it holds for $n + 1$.

From the Envelope condition we know that at the solution $a_{n+1}'(a, \varepsilon)$ corresponding to $Tv_n(a, \varepsilon)$ we have:

$$v_{n+1}'(a, \varepsilon) = u'(\varepsilon + a - qa_{n+1}'(a, \varepsilon))$$

From the Euler equation evaluated at some a':

$$u'(a+\varepsilon-qa') \geq (\beta/q)\sum_{\varepsilon'}\pi(\varepsilon'|\varepsilon)v_n'(a',\varepsilon') \equiv (\beta/q)\psi(a',\varepsilon) \quad \text{w.e. if } a' > \underline{a}$$

and from the assumption of monotonicity on π:

$$\pi(\bar{\varepsilon}|\bar{\varepsilon}) \geq \pi(\bar{\varepsilon}|\underline{\varepsilon})$$

$$1-\pi(\bar{\varepsilon}|\bar{\varepsilon}) = \pi(\underline{\varepsilon}|\bar{\varepsilon}) \leq \pi(\underline{\varepsilon}|\underline{\varepsilon}) = 1-\pi(\bar{\varepsilon}|\underline{\varepsilon})$$

we know that $\psi(a',\bar{\varepsilon}) \geq \psi(a',\underline{\varepsilon})$. To see this take the difference:

$$\begin{aligned}
\psi(a',\underline{\varepsilon}) - \psi(a',\bar{\varepsilon}) &= \Big(\pi(\underline{\varepsilon}|\underline{\varepsilon}) - \pi(\underline{\varepsilon}|\bar{\varepsilon})\Big)v_n'(a',\underline{\varepsilon}) - \Big(\pi(\bar{\varepsilon}|\bar{\varepsilon}) - \pi(\bar{\varepsilon}|\underline{\varepsilon})\Big)v_n'(a',\bar{\varepsilon}) \\
&= \Big((1-\pi(\underline{\varepsilon}|\underline{\varepsilon})) - (1-\pi(\bar{\varepsilon}|\bar{\varepsilon}))\Big)v_n'(a',\underline{\varepsilon}) - \Big(\pi(\bar{\varepsilon}|\bar{\varepsilon}) - \pi(\bar{\varepsilon}|\underline{\varepsilon})\Big)v_n'(a',\bar{\varepsilon}) \\
&= \Big(\pi(\bar{\varepsilon}|\bar{\varepsilon}) - \pi(\bar{\varepsilon}|\underline{\varepsilon})\Big)\Big(v_n'(a',\underline{\varepsilon}) - v_n'(a',\bar{\varepsilon})\Big)
\end{aligned}$$

we know that by assumption on π we know that $\pi(\bar{\varepsilon}|\bar{\varepsilon}) \geq \pi(\bar{\varepsilon}|\underline{\varepsilon})$ and by induction we know that $v_n'(a',\underline{\varepsilon}) \geq v_n'(a',\bar{\varepsilon})$. Lastly, the induction step is to notice that for a such that $a' = a_{n+1}(a,\varepsilon) > \underline{a}$ we have:

$$\begin{aligned}
u'(a+\underline{\varepsilon}-qa_{n+1}'(a,\underline{\varepsilon})) &= (\beta/q)\psi(a_{n+1}'(a,\underline{\varepsilon}),\underline{\varepsilon}) \\
&\geq (\beta/q)\psi(a_{n+1}'(a,\underline{\varepsilon}),\bar{\varepsilon}) \\
&> (\beta/q)\psi(a_{n+1}'(a,\bar{\varepsilon}),\bar{\varepsilon}) = u'(a+\bar{\varepsilon}-qa_{n+1}'(a,\bar{\varepsilon}))
\end{aligned}$$

where the inequality in the second line follows from $\psi(a',\underline{\varepsilon}) \geq \psi(a',\bar{\varepsilon})$ and the inequality in the third lines follows from Lemma 0, $a'(a,\bar{\varepsilon}) > a'(a,\underline{\varepsilon})$, and the concavity of v. So then:

$$u'(a+\underline{\varepsilon}-qa_{n+1}'(a,\underline{\varepsilon})) = v_{n+1}'(a,\underline{\varepsilon}) \geq v_{n+1}'(a,\bar{\varepsilon}) = u'(a+\bar{\varepsilon}-qa_{n+1}'(a,\bar{\varepsilon}))$$

thereby completing the induction proof.

SLP lemma 3.7 ensures that the decision rule converges pointwise, $a_n'(a,\varepsilon) \to a'(a,\varepsilon)$ for each (a,ε). But then so does $v_n'(a,\varepsilon) \to v'(a,\varepsilon)$ since $v_n'(a,\varepsilon) = u'(a+\varepsilon-qa_n'(a,\varepsilon))$. Therefore, $v'(a,\bar{\varepsilon}) \leq v'(a,\underline{\varepsilon})$. ■

Lemma 2: If $v'(a,\varepsilon) > (\beta/q)\mathbb{E}[v'(a,\varepsilon')|\varepsilon]$ for $a \geq a^* > \underline{a}$, then $a'(a,\varepsilon) < a$ for all $a \geq a^*$.

Proof: Take $a \geq a^* > \underline{a}$. If $a'(a,\varepsilon) = \underline{a}$ then since $a > \underline{a}$ we have $a'(a,\varepsilon) = \underline{a} < a$ by assumption.

On the other hand, if $a'(a,\varepsilon) > \underline{a}$ then strict concavity of v and u gives $a'(a,\varepsilon) < a$. To see this rewrite the Euler equation:

$$v'(a,\varepsilon) = (\beta/q)\sum_{\varepsilon'} \pi(\varepsilon'|\varepsilon) v'(a'(a,\varepsilon),\varepsilon')$$

But by hypothesis:

$$v'(a,\varepsilon) > (\beta/q)\sum_{\varepsilon'} \pi(\varepsilon'|\varepsilon) v'(a,\varepsilon')$$

Since $v'(a,\varepsilon')$ is decreasing in a, the value for $a'(a,\varepsilon)$ must be lower than a on the RHS of the above equation so that the Euler equation holds. Hence, $a'(a,\varepsilon) < a$. ■

I will now show, in Lemmas 3 and 4, that the hypothesis of Lemma 2 is verified for $\varepsilon = \underline{\varepsilon}$ but not for $\varepsilon = \overline{\varepsilon}$.

Lemma 3: $a(a,\underline{\varepsilon}) < a$ for all $a > \underline{a}$.

Proof: The result follows from the assumption $\beta < q$ (the first inequality) and from Lemma 1 (the second inequality) that $v'(a,\overline{\varepsilon}) \leq v'(a,\underline{\varepsilon})$:

$$v'(a,\underline{\varepsilon}) \;>\; (\beta/q)v'(a,\underline{\varepsilon}) \;\geq\; (\beta/q)v'(a,\underline{\varepsilon}) - (\beta/q)\pi(\overline{\varepsilon}|\underline{\varepsilon})\Big(\underbrace{v'(a',\underline{\varepsilon}) - v'(a',\overline{\varepsilon})}_{\geq 0 \;\forall\; a'}\Big)$$

Therefore by Lemma 2, $a(a,\underline{\varepsilon}) < a$ for all $a > \underline{a}$. ■

Lemma 4: There exists a such that $a'(a,\overline{\varepsilon}) = a$.

Proof: Suppose for contradiction that $a'(a,\overline{\varepsilon}) > a$ for all a. We want to show that this assumption leads to the conclusion that:

$$v'(a,\overline{\varepsilon}) > (\beta/q)\sum_{\varepsilon'} \pi(\varepsilon'|\overline{\varepsilon}) v'(a,\varepsilon') \quad \forall\; a \geq a^* > \underline{a}$$

which implies $a'(a,\overline{\varepsilon}) < a$, by Lemma 2.

Suppose for contradiction that $a(a,\overline{\varepsilon}) > a$ for all a. In *Lemma 3* we showed that $a'(a,\underline{\varepsilon}) < a$ for all $a > \underline{a}$. Therefore, by assumption and Lemma 3: $a'(a,\overline{\varepsilon}) \geq a'(a,\underline{\varepsilon})$ *for all* a. Then for all a:

$$c(a,\underline{\varepsilon}) = a + \underline{\varepsilon} - qa'(a,\underline{\varepsilon}) \geq a + \overline{\varepsilon} - qa'(a,\overline{\varepsilon}) = c(a,\overline{\varepsilon})$$

$$c(a,\underline{\varepsilon}) \geq c(a,\overline{\varepsilon}) - (\overline{\varepsilon} - \underline{\varepsilon})$$

$$\frac{c(a,\underline{\varepsilon})}{c(a,\overline{\varepsilon})} \geq 1 - \frac{\overline{\varepsilon} - \underline{\varepsilon}}{c(a,\overline{\varepsilon})}$$

From assumption that $a'(a,\overline{\varepsilon}) > a$ for all a, we know that as $a \to \infty$, $c(a,\overline{\varepsilon}) > \overline{\varepsilon} + (1-q)a \to \infty$. Then, since $u'(c) = c^{-\sigma}$, for a sufficiently large (so that $c(a,\overline{\varepsilon})$ sufficiently large):

$$\frac{v'(a,\overline{\varepsilon})}{v'(a,\underline{\varepsilon})} = \left(\frac{c(a,\underline{\varepsilon})}{c(a,\overline{\varepsilon})}\right)^{\sigma} \geq \left(1 - \frac{\overline{\varepsilon} - \underline{\varepsilon}}{c(a,\overline{\varepsilon})}\right)^{\sigma}$$

Then since $\beta/q < 1$, there exists a^* such that for all $a \geq a^*$ we have:

$$\frac{v'(a,\overline{\varepsilon})}{v'(a,\underline{\varepsilon})} \geq \left(1 - \frac{\overline{\varepsilon} - \underline{\varepsilon}}{c(a,\overline{\varepsilon})}\right)^{\sigma} > (\beta/q)$$

Therefore:

$$v'(a,\overline{\varepsilon}) > (\beta/q)v'(a,\underline{\varepsilon}) \geq (\beta/q)\Big(v'(a,\underline{\varepsilon}) - \pi(\overline{\varepsilon}|\overline{\varepsilon})\Big[v'(a,\underline{\varepsilon}) - v'(a,\overline{\varepsilon})\Big]\Big)$$

where the second line follows from Lemma 1, $v'(a,\underline{\varepsilon}) \geq v'(a,\overline{\varepsilon})$. But then for all $a \geq a^*$ the hypothesis of Lemma 2 holds thus yielding the contradiction. ■

Lemmas 1-4 imply that there exists an ergodic set $S = [\underline{a},\overline{a}] \times \{\underline{\varepsilon},\overline{\varepsilon}\}$. Lemma 4 shows that there is a fixed point of $a'(a,\overline{\varepsilon}) = a$, and we can then choose $\overline{\alpha}$ to be the smallest such fixed point. Lemma 3 shows that $a'(a,\underline{\varepsilon}) < a$ for all $a > \underline{a}$.

Second Step (Stationary Distribution): Now I will proceed to the second step. Show that the following conditions for Hopenhayn and Prescott's (1987) theorem 2 hold:

Theorem 2 (HP87): If Assumptions 1-5 hold

1. $(S, \geq)$ is an ordered space
2. S is a compact metric space
3. $\geq$ is a closed order
4. $(S, \mathscr{B})$ is a measurable space and $\mathscr{B}$ is the Borel σ-algebra

5. $P : S \times \mathscr{B} \to [0,1]$ is a transition function

and furthermore if P is increasing, S has greatest and least elements $(c,d) \in S$ respectively and the Monotone Mixing Condition holds, then there exists a unique probability measure λ and $T^*\lambda_0 \to \lambda$ for any any $\lambda_0 \in M(S)$.

The Monotone Mixing Condition states: There exists $s^* \in S$, $\epsilon > 0$, and N such that $P^N(d, \{s \mid s \leq s^*\}) > \epsilon$ and $P^N(c, \{s \mid s \geq s^*\}) > \epsilon$.

First, define an order $\geq$ on S. For $x, x' \in S$ where $x = (x_1, x_2)$, $x \geq x'$ iff ($x_1 \geq x_1'$ and $x_2 = x_2'$) or ($x' = c = (\underline{a}, \underline{\varepsilon})$) or ($x = d = (\overline{a}, \overline{\varepsilon})$). This order is closed with minimum and maximum elements (c,d).

Next define the transition function P as before and show that it is increasing. Hopenhayn and Prescott (1987) prove that it is sufficient to show that

$$x, x' \in S,\ x \geq x' \qquad \Longrightarrow \qquad \int_S \mathbf{1}[x \in B] P(x, dx) \geq \int_S \mathbf{1}[x \in B] P(x', dx)$$

where $B = \{y \in S \mid y \geq x \text{ for some } x \in B\} \in \mathscr{B}$.

Proof: Let $B_x = \{z \in Z \mid g(x,z) \in B\}$ and $B_{x'} = \{z \in Z \mid g(x',z) \in B\}$. Show that $B_{x'} \subset B_x$ for $x \geq x'$. This is obvious if $g(x,z)$ is monotone in x for fixed z. It is straightforward but tedious to show. Therefore $P(x,B) \geq P(x',B)$. ■

Lastly, show that the Monotone Mixing Condition is satisfied.

Proof: Choose $s^* = \{\frac{1}{2}(a(\underline{a}, \overline{\varepsilon}) + \overline{a}), \overline{\varepsilon}\}$. Define a sequence $\{x_i\}_{i=1}^{\infty}$ such that $x_1 = \underline{a}$ and $x_i = a'(x_{i-1}, \overline{\varepsilon})$ for all $i > 1$. Define a sequence $\{y_i\}_{i=1}^{\infty}$ such that $t_1 = \overline{a}$ and $y_i = a'(y_{i-1}, \underline{\varepsilon})$ for all $i > 1$. These sequences *monotonically* converge to $(\overline{a}, \underline{a})$ respectively. Therefore there exists an N_x such that for all $n \geq N_x$, $Pr\{x_n \in S \mid x_n \geq s^*\} > 0$ given $x_1 = (\underline{a}, \underline{\varepsilon}) = c$. Likewise there exists an N_y such that for all $n \geq N_y$, $Pr\{y_n \in S \mid y_n \leq s^*\} > 0$ given $y_1 = (\overline{a}, \overline{\varepsilon}) = d$. Therefore choose $N = \max\{N_x, N_y\}$ in the Mixing Condition. ■

Existence of Equilibrium: This section will resume analysis of the production economy (c.f. Aiyagari (1994)) from section 4.1 as a reference. An equilibrium coincides with a price $1 + r \equiv 1/q$ that clears markets:

$$K(r) = \int_{A \times E} a'(a, \varepsilon; r) \lambda^*(da, d\varepsilon; r) \equiv A(r)$$

There exists a unique r that clears markets if (i) (K, A) are continuous functions of r, (ii)

$K(r)$ and $A(r)$ are monotonic but in different directions, and (iii) the limits of $K(r)$ and $A(r)$ with respect to r are such that $\{r \mid (K(r) < M) \wedge (A(r) < M)\} \neq \emptyset$ for M large.

In the discussion of Huggett (1993) we have already showed that there exists a unique $\lambda^*(r)$ and furthermore that a' is the optimal policy function. Hence we have shown that $A(r)$ exists and in fact that it is continuous.

In the Aiyagari economy, we know that $K(r) = (\alpha/(r+\delta))^{1/(1-\alpha)}L$. Therefore $K'(r) < 0$ is strictly monotonic decreasing. However, there exist no results on the monotonicity of $A(r)$ with respect to r, so uniqueness is not guaranteed.

Lastly, we must show $K(r)$ and $A(r)$ have limits. On the demand side, $K(r) \to 0$ as $r \to \infty$ and $K(r) \to \infty$ as $r \to -\delta$. Therefore the codomain is given by $K([-\delta, \infty)) \in [0, \infty)$. On the supply side, properties of $A(r)$ are more complicated because we must analyze how agents' optimality conditions respond to limiting values of r. Suppose that $\underline{a} = 0$. We know that when $r = -\delta < 0$ then a unit of savings generates negative interest, in which case an agent has fewer incentives to save. In fact, if $r = -1$ then savings generates zero return and the agent never saves. On the other hand if $1 + r \geq 1/\beta$ then the agent's Euler equation is a supermartingale and asset accumulation goes to infinity in the long run.

Hence, there exist values of r for which the excess demand function is greater than and less than zero:

$$\lim_{r \to (1/\beta - 1)} \Big(K(r) - A(r)\Big) = -\infty$$

$$\lim_{r \to -\delta} \Big(K(r) - A(r)\Big) = \infty$$

By continuity, there exists a value for r such that excess demand is zero.

4.3 Computation

Although theory provides us with proofs of existence, we must rely on computation in order to further characterize this economy's allocation and prices. In this section we first calibrate the model. Then will outline methods for computing (i) the agent's policy functions and value function (v, c, a'), (ii) the stationary distribution over wealth and income shocks λ, and (iii) equilibrium prices (r, w).

Calibration We must assign values to model parameters in order to compute equilibrium allocations and prices. To start, we specify a standard Cobb-Douglas production function:

$$F(K, L) = K^{\alpha} L^{1-\alpha}$$

where $\alpha = 1/3$ is chosen to equal capital income's share of total income. We assume that capital depreciates at a rate of six percent, $\delta = 0.06$.

Turning to preferences, we specify CRRA utility function:

$$u(c) = \frac{c^{1-\sigma}}{1-\sigma}$$

The parameter σ is the coefficient of relative risk aversion and usually is assigned a value between 1 and 6. Here, we choose a standard value of $\sigma = 3$. The other preference parameter, the discount factor β, is chosen according to long-run restrictions on the model. In particular, the parameter β should be chosen to match a capital-output ratio of 3. While this can be done using a method of moments procedure (e.g. rougly speaking, recomputing the model until you find a β value that is consistent with $K/Y = 3$), a useful approximation is to consider the complete markets steady-state case in which we know:

$$\begin{aligned} 1 &= \beta \left(\alpha K^{\alpha-1} L^{1-\alpha} + (1-\delta) \right) \\ \beta &= \frac{1}{1 + \alpha \frac{Y}{K} - \delta} = \frac{1}{1 + (1/3)(1/3) - 0.06} \approx 0.9514 \end{aligned}$$

With incomplete markets you shoud set β to be slightly less than 0.9514 since the precautionary savings motive will generate a larger capital-output ratio than 3 for the same value of β.

Next consider the labor income process, which is represented as an AR1 process:

$$\log(\varepsilon_t) = (1-\rho)\mu + \rho \log(\varepsilon_{t-1}) + \epsilon_t \quad \text{s.t. } \epsilon_t \sim \mathcal{N}(0, \sigma_{\epsilon}^2)$$

Using the Panel Study of Income Dynamics (PSID), typical estimates are a scale parameter of $\mu = 0$, a autocorrelation coefficient of $\rho = 0.95$, and a standard deviation of innovations $\sigma_{\epsilon} = 0.20$. This is considered a very simple income process specification, and it is often enhanced with additional features that better fit the available data.

Lastly, the borrowing constraint parameter $\underline{a}$ could be chosen to help the model match features of the wealth distribution, such as the fraction of individuals with negative wealth. To do this, however, would require searching over model parameters until you find the value for which the model corresponds to the desired feature of the data. As a first pass, a simpler approach is to study a "no borrowing constraint," $\underline{a} = 0$.

Therefore there are 7 parameters with values presented in table 6.

Table 6: Baseline Parameters

β	σ	α	δ	ρ	σ_ϵ	$\underline{a}$
0.94	3.0	0.33	0.06	0.95	0.20	0.0

Markov Chain Discretization: In order to compute equilibrium in a tractable way, we will discretize the continuous income process given by:

$$y_t = \rho y_{t-1} + \epsilon_t \quad \text{s.t. } \epsilon_t \sim \mathcal{N}(0, \sigma_\epsilon^2)$$

where $y_t \equiv \log(\varepsilon_t)$ and $\rho \in (0, 1)$. A discretization requires a method for choosing n discrete points for $\{y_1, y_2, \ldots, y_n\}$ and choosing transition probabilities. There are several good methods, and I particularly recommend that you read Kopecky and Suen's (2010) work on the Rouwenhorst method. In these notes I will describe Tauchen's (1986) method, which is a benchmark method for discretization.

To choose the n discrete points $\{y_1, y_2, \ldots, y_n\}$, Tauchen (1986) recommends a symmetric grid with $y_1 = -y_n$ such that y_n is three standard deviations from the mean (zero):

$$y_n = 3 \cdot \frac{\sigma_\epsilon}{\sqrt{1 - \rho^2}}$$

Let the remaining $n - 2$ points $\{y_2, \ldots, y_{n-1}\}$ be equally spaced on the interval $[y_1, y_n]$ such that:

$$y_{i+1} = y_i + \Delta \quad \forall \ i = 1, \ldots, n - 1$$

which recovers the step size Δ from $i = n - 1$ as:

$$\Delta = \frac{y_n - y_1}{n - 1}$$

Denote the transition probability from node i to node j as $\pi(y_j|y_i)$. To choose this transition probability, we will partition the the interval $[y_1, y_n]$ into non-overlapping intervals. For each j, we will compute the probability over the interval $[y_j - (1/2)\Delta, y_j + (1/2)\Delta]$ by recalling that (i) if $y_{t-1} = y_i$ then $y_t = y_j$ only if $\epsilon_t = y_j - \rho y_i$ and (ii) $\epsilon \sim \mathcal{N}(0, \sigma_\epsilon^2)$. Accordingly the transition probabilities for all $j = 2, \ldots, n-1$ are derived below:

$$\begin{aligned}
\pi(y_j|y_i) &= \text{Prob}\Big(y_j \mid \tilde{y}_{t-1} = y_i\Big) \\
&= \text{Prob}\Big(y_j - \frac{1}{2}\Delta < \rho y_i + \epsilon_t \le y_j + \frac{1}{2}\Delta\Big) \\
&= \text{Prob}\Big(y_j - \rho y_i - \frac{1}{2}\Delta < \epsilon_t \le y_j - \rho y_i + \frac{1}{2}\Delta\Big) \\
&= \Phi\left(\frac{y_j - \rho y_i + \frac{1}{2}\Delta}{\sigma_\epsilon}\right) - \Phi\left(\frac{y_j - \rho y_i - \frac{1}{2}\Delta}{\sigma_\epsilon}\right)
\end{aligned}$$

For $j = 1$, we assume that there is zero mass below $y_1 - \rho y_i$ so that:

$$\pi(y_1|y_i) = \Phi\left(\frac{y_1 - \rho y_i + \frac{1}{2}\Delta}{\sigma_\epsilon}\right)$$

For $j = n$, we assume that all mass lies below $y_n + \rho y_i$ so that:

$$\pi(y_n|y_i) = 1 - \Phi\left(\frac{y_n - \rho y_i - \frac{1}{2}\Delta}{\sigma_\epsilon}\right)$$

where $\Phi(\cdot)$ is the CDF of a standard normal random variable.

Lastly, to choose the number of grid points n, compute the desired statistics from the discrete Markov chain and compare them to the exact (theoretical) statistics of the autoregressive process. One way to compute statistics from the discrete Markov chain is to draw a large number, say T, of innovations $\{\varepsilon_t\}_{t=1}^T$ and simulate the process $\{\hat{y}_t\}_{t=1}^T$ given $y_0 = 0$ by using $\pi(y_j|y_i)$.

For example, the serial correlation and unconditional variance are given by:

$$\hat{\rho} = \frac{cov(\hat{y}_t, \hat{y}_{t-1})}{var(\hat{y}_t)} \quad \text{and} \quad \hat{\sigma}_\epsilon^2 = (1 - \hat{\rho}^2) var(\hat{y}_t)$$

where $\hat{\rho}$ and $\hat{\sigma}_\epsilon$ are estimated from $\{\hat{y}_t\}_{t=1}^T$. Usually one chooses $T = 11,000$ and only computes statistics after discarding the first 1000 values of $\hat{y}_t$. If the estimated values are "too different" from the theoretical values (ρ, σ_ϵ) then choose a larger n. Notice that choosing a larger n is equivalent to choosing a smaller Δ. As $n \to \infty$ then $\Delta \to 0$ and we recover the continuous autoregressive process.

Note that the Tauchen (1986) method is known to provide poor approximations to highly autocorrelated processes. In those cases, Rouwenhorst's method has proven to provide a better fit (see Kopecky and Suen (2010) for an exposition and diagnostics).

Preliminaries and Notation: We have already shown how to construct a discrete grid governing income shocks, $\varepsilon_i = \exp(y_i)$. We now must also construct a discretization of the state space for assets. Denote the asset and income grids by $\mathbb{G}_a \equiv \{\underline{a}, \ldots, \bar{a}\}$ and $\mathbb{G}_\varepsilon \equiv \{\varepsilon_1, \ldots, \varepsilon_{n_\varepsilon}\}$, respectively. Denote the number of grid points in the asset grid by n_a and in the income grid by n_ε.

When the consumption policy function is concave, we can improve accuracy by placing more asset grid points near the lower bound $\underline{a}$ since this will be the region in which the policy function is the most non-linear and has the steepest slope. In this region near the lower bound, a small change in wealth induces a large change in the consumption policy function. Near the upper bound $\bar{a}$, at high levels of wealth, the policy function will be nearly linear and relatively flat. An small change in wealth does not induce a large change in consumption near the upper bound. Therefore, in this upper region additional grid points will not improve numerical accuracy to a "large" degree. Furthermore, with the borrowing constraint, savings policy functions may have kinks in the region of the constraint (here the lower bound $\underline{a}$). Again, placing more grid points near the anticipated kink will improve accuracy of the solution. Following Maliar, Maliar and Valli (2010), consider the following formula for choosing asset grid points:

$$a_i = \underline{a} + \left(\frac{i-1}{n_a - 1}\right)^\theta (\bar{a} - \underline{a}) \qquad \forall\, i = 1, \ldots, n_a$$

The parameter θ determines the relative placement of grid points near the lower bound $\underline{a}$. When $\theta = 1$ then the grid has equally spaced nodes on the interval $[\underline{a}, \bar{a}]$. When $\theta > 1$, more grid points are concentrated near the lower bound. When $\theta < 1$ more grid points are concentrated near the upper bound. Finding an appropriate value for θ depends on the particular model and application, and therefore may require some experimentation.

Next, I will make heavy use of linear interpolation in the following computational methods. Modern methods tend to circumvent costly root-finding procedures in exchange for cheap interpolation methods. Suppose we have a grid $\mathbb{G}_x$ and associated functional values of $f(x_i)$ for each point $x_i \in \mathbb{G}_x$ on the grid. Now suppose we wish to approximate the value of f for some point not on the grid, $\hat{x} \notin \mathbb{G}_x$. Linear interpolation computes such an approximation by finding i such that $x_i \leq \hat{x} \leq x_{i+1}$ and computing:

$$f(\hat{x}) = \left(1 - \frac{\hat{x} - x_i}{x_{i+1} - x_i}\right) f(x_i) + \frac{\hat{x} - x_i}{x_{i+1} - x_i} f(x_{i+1})$$

For convenience, denote the interpolation of $\hat{x}$ onto a pair $x_i \in \mathbb{G}_x$ and $f(x_i)$ as:

$$f(\hat{x}) = \mathcal{I}\Big(x \in \mathbb{G}_x, f(x) \;\Big|\; \hat{x}\Big)$$

Lastly, computational methods tend to be iterative. This means that starting from an initial guess, the method updates the guess and re-iterates from the updated guess. For a policy function, say for consumption, $c_i(a, \varepsilon)$ denotes the i-th iteration of the policy function defined on grid points $(a, \varepsilon) \in \mathbb{G}_a \times \mathbb{G}_\varepsilon$.

Agent's Dynamic Program: We will look at two general techniques for globally solving the agent's recursive problem. The first is a version of fixed point iteration from Maliar and Maliar's (2013) Envelope Condition Method. The second is a version of time iteration from Carroll's (2006) Endogenous Gridpoint Method. While I will demonstrate these methods using policy function iteration, they easily generalize to accommodate value function iteration (see Maliar and Maliar (2013) and Barillas and Fernández-Villaverde (2007)). Furthermore see Heer and Maussner (2009) for greater detail on computational methods relevant to this economic environment.

To reiterate, we will find the policy functions $c(a, \varepsilon)$ and $a'(a, \varepsilon)$ of the agent's dynamic program:

$$v(a, \varepsilon) = \max_{c, a'} u(c) + \beta \sum_{\varepsilon' \in E} \pi(\varepsilon' | \varepsilon) v(a', \varepsilon')$$

$$\text{s.t.} \quad c + a' \leq w\varepsilon + (1 + r)a$$

$$a' \geq 0$$

with associated Euler equation:

$$u_c(c) \geq \beta(1 + r) \sum_{\varepsilon \in E} \pi(\varepsilon' | \varepsilon) u_c(c') \quad \text{w.e. if } a' > 0$$

where the Euler equation holds with equality when the borrowing constraint is slack. Furthermore assume the utility has the CRRA functional form $u(c) = c^{1-\sigma}/(1 - \sigma)$ such that

$u_c(c) = c^{-\sigma}$. In what follows we take the prices (r, w) and all parameters as given.

We will start with **Fixed Point Iteration:**

1. Guess a policy function for savings, $a'_0(a, \varepsilon)$ for all $(a, \varepsilon) \in \mathbb{G}_a \times \mathbb{G}_\varepsilon$

2. Construct the right hand side of the Euler equation:

$$B_i(a, \varepsilon) \equiv \beta(1+r) \sum_{\varepsilon' \in \mathbb{G}_\varepsilon} \pi(\varepsilon'|\varepsilon) c'(a, \varepsilon, \varepsilon')^{-\sigma}$$

   where tomorrow's consumption is computed by interpolation:

$$c'(a, \varepsilon, \varepsilon') = w\varepsilon' + (1+r)a'_i(a, \varepsilon) - a''(a, \varepsilon, \varepsilon')$$

   where the last term is the savings choice in the next period, given by:

$$a''(a, \varepsilon, \varepsilon') = \mathcal{I}\left(a \in \mathbb{G}_a, a'_i(a, \varepsilon') \;\middle|\; a'_i(a, \varepsilon)\right)$$

   Impose $c'(a, \varepsilon, \varepsilon') \geq 0$ if violated.

3. Compute an updated policy function for consumption from the Euler equation:

$$c_{i+1}(a, \varepsilon) = B_i(a, \varepsilon)^{-\frac{1}{\sigma}}$$

   and impose $c_{i+1}(a, \varepsilon) \leq w\varepsilon + (1+r)a - \underline{a}$ if violated.

4. Compute an updated policy function for savings from the budget constraint:

$$\hat{a}'_{i+1}(a, \varepsilon) = w\varepsilon + (1+r) - c_{i+1}(a, \varepsilon)$$

   and impose $\hat{a}'_{i+1}(a, \varepsilon) \geq \underline{a}$ if violated.[8]

5. Update the savings policy function by dampening with some fixed $\zeta \in (0, 1)$:

$$a'_{i+1}(a, \varepsilon) = (1-\zeta)a'_i(a, \varepsilon) + \zeta \hat{a}'_{i+1}(a, \varepsilon)$$

6. If $a'_{i+1} = a'_i$ up to the desired tolerance, then stop.

[8]The borrowing constraint is already guaranteed by imposing the upper bound on $c_{i+1}(a, \varepsilon)$.

More specifically: stop if the following holds for a fixed $\kappa \in \mathbb{N}_{++}$

$$\max_{(a,\varepsilon)\in\mathbb{G}_a\times\mathbb{G}_\varepsilon} \left| \frac{a'_{i+1}(a,\varepsilon) - a'_i(a,\varepsilon)}{1 + a'_i(a,\varepsilon)} \right| < 10^{-\kappa}$$

Otherwise go back to step 2 with $a'_i \to a'_{i+1}$.

Next we will solve for policy functions using the **Endogenous Gridpoint Method**. This method will use a clever change of variables to speed up convergence. In particular, let the state space be defined over tomorrow's wealth instead of today's wealth. Accordingly, the state space is $(a', \varepsilon) \in \mathbb{G}_a \times \mathbb{G}_\varepsilon$.

1. Guess a policy function for tomorrow's savings, $a'_0(a', \varepsilon)$ for all $(a', \varepsilon) \in \mathbb{G}_a \times \mathbb{G}_\varepsilon$

2. Construct the right hand side of the Euler equation:

$$B_i(a', \varepsilon) \equiv \beta(1 + r) \sum_{\varepsilon' \in \mathbb{G}_\varepsilon} \pi(\varepsilon'|\varepsilon) c'(a', \varepsilon, \varepsilon')^{-\sigma}$$

   where tomorrow's consumption is computed by interpolation:

$$c'(a', \varepsilon, \varepsilon') = w\varepsilon' + (1 + r)a' - a'_i(a', \varepsilon')$$

   Impose $c'(a', \varepsilon, \varepsilon') \geq 0$ if violated.

3. Compute an updated policy function for consumption from the Euler equation:

$$c_{i+1}(a', \varepsilon) = B_i(a', \varepsilon)^{-\frac{1}{\sigma}}$$

4. Using the budget constraint, compute the quantity of savings that the agent must have entered the period with in order to have chosen to save $a' \in \mathbb{G}_a$ for tomorrow:

$$\hat{a}_{i+1}(a', \varepsilon) = \frac{1}{1 + r} \Big(c_{i+1}(a', \varepsilon) + a' - w\varepsilon \Big)$$

   This is the **endogenous grid** over assets.

5. Update the savings policy function by using the endogenous grid to interpolate savings back onto next period's asset grid $\mathbb{G}_a$:

$$a'_{i+1}(a', \varepsilon') = \mathcal{I}\Big(\hat{a}_{i+1}(a', \varepsilon'), a' \in \mathbb{G}_a \;\Big|\; a' \in \mathbb{G}_a \Big)$$

and impose $a'_{i+1}(a', \varepsilon') \geq \underline{a}$ if violated.

6. If $a'_{i+1} = a'_i$ up to the desired tolerance, then stop.
   More specifically: stop if the following holds for a fixed $\kappa \in \mathbb{N}_{++}$

$$\max_{(a',\varepsilon) \in \mathbb{G}_a \times \mathbb{G}_\varepsilon} \left| \frac{a'_{i+1}(a', \varepsilon) - a'_i(a', \varepsilon)}{1 + a'_i(a', \varepsilon)} \right| < 10^{-\kappa}$$

   Otherwise go back to step 2 with $a'_i \to a'_{i+1}$.

Speed and accuracy benchmarks for the two methods can be found in Maliar and Maliar (2013). Generally, the two methods are very comparable for incomplete market models. However, if the budget constraint is non-linear in state variables then Fixed Point Iteration is faster than the Endogenous Gridpoint Method. This is because the Endogenous Gridpoint Method requires a costly rootfinding procedure for each (a, ε) at the final step to recover the endogenous grid.[9] Fixed Point Iteration, on the other hand, does not require such a step. Fixed Point Iteration, however, does not guarantee convergence and may be more prone to numerical instability depending on the application.

Stationary Distributions: I follow Young (2010) and Heer and Maussner (2009) (Chapter 5.2, section on distribution function iteration) in computing the stationary distribution.

The basic strategy is to define the distribution as a histogram, so it has mass on each gridpoint (a, ε). Then use the savings policy function to compute the distribution over the level of savings individuals will hold at the end of the period. This transition is governed by:

$$P\left((a, \varepsilon), (a', \varepsilon')\right) = \pi(\varepsilon'|\varepsilon) \mathbb{1}\left[a'(a, \varepsilon) = a'\right]$$

Since a is defined on gridpoints $\mathbb{G}_a$, savings $a'(a, \varepsilon)$ will generically not be on the grid $\mathbb{G}_a$. To force mass onto gridpoints, reassign the mass $\lambda(a, \varepsilon)$ to the nodes given by $a_i \leq a'(a, \varepsilon) \leq a_{i+1}$ according to a simple lottery, with weights ω_i given below:

$$(1 - \omega_i)a_i + \omega_i a_{i+1} = a'(a, \varepsilon) \quad \Longrightarrow \quad \omega_i = \frac{a'(a, \varepsilon) - a_i}{a_{i+1} - a_i}$$

Lastly iterate on the distribution using the T^* operator to find a fixed point over distributions

[9] To illustrate this point, consider the representative agent Neoclassical Growth Model with production technology $f(k, z) = zk^\alpha$. To compute the consumption and investment policy functions using endogenous gridpoints requires a change of variables to cash-in-hand, $y(k, z) = zk^\alpha + (1 - \delta)k$ which is updated as $\hat{y}(k', z) = c(k', z) + k'$. Once the policy function is computed, it is necessary to find the endogenous grid on capital instead of cash-in-hand. This requires a root-finder since k^α introduces a non-linearity.

through iteration.

$$(T^*\lambda)(\mathcal{A}\times\mathcal{E}) = \int_{A\times E} P\left((a,\varepsilon),\mathcal{A}\times\mathcal{E}\right)d\lambda \qquad \forall\ \mathcal{A}\times\mathcal{E}\in\mathscr{B}$$

Below is the pseudo-algorithm for computing the steady state distribution:

1. Construct a new grid for assets $\mathbb{G}_a^\lambda$ with more gridpoints than $\mathbb{G}_a$, such that $n_a^\lambda > n_a$ and is uniformly spaced ($\theta = 1$).

2. Guess an initial distribution function, $\lambda_0(a,\varepsilon)$ for all $(a,\varepsilon)\in\mathbb{G}_a^\lambda\times\mathbb{G}_\varepsilon$

3. Compute weights for each $j = 1,\ldots,n_a^\lambda$ and $(a,\varepsilon)\in\mathbb{G}_a^\lambda\times\mathbb{G}_\varepsilon$ such that:

   - For all $j = 2,\ldots,n_a^\lambda - 1$:

$$\omega_j(a,\varepsilon) = \begin{cases} 1-\dfrac{a'(a,\varepsilon)-a_j}{a_{j+1}-a_j} & \text{if } a_{j-1}\le a'(a,\varepsilon)\le a_j \\ \dfrac{a'(a,\varepsilon)-a_j}{a_{j+1}-a_j} & \text{if } a_j\le a'(a,\varepsilon)\le a_{j+1} \\ 0 & \text{otherwise} \end{cases}$$

   - For $j = 1$:

$$\omega_1(a,\varepsilon) = \begin{cases} 1-\dfrac{a'(a,\varepsilon)-a_1}{a_2-a_1} & \text{if } a_1\le a'(a,\varepsilon)\le a_2 \\ 1 & \text{if } a'(a,\varepsilon) < a_1 \\ 0 & \text{otherwise} \end{cases}$$

   - For $j = n_a^\lambda$:

$$\omega_{n_a^\lambda}(a,\varepsilon) = \begin{cases} 1 & \text{if } a'(a,\varepsilon) > a_{n_a^\lambda} \\ \dfrac{a'(a,\varepsilon)-a_{n_a^\lambda-1}}{a_{n_a^\lambda}-a_{n_a^\lambda-1}} & \text{if } a_{n_a^\lambda-1}\le a'(a,\varepsilon)\le a_{n_a^\lambda} \\ 0 & \text{otherwise} \end{cases}$$

4. Update the distribution using the transition function for all $j = 1,\ldots,n_a^\lambda$, given $\lambda_i(a,\varepsilon)$:

$$\lambda_{i+1}(a_j,\varepsilon') = \sum_{\varepsilon\in\mathbb{G}_\varepsilon}\pi(\varepsilon'|\varepsilon)\sum_{a\in\mathbb{G}_a^\lambda}\omega_j(a,\varepsilon)\lambda_j(a,\varepsilon)$$

5. If $\lambda_{i+1} = \lambda_i$ up to the desired tolerance, then stop.
   That is, stop if the following holds for fixed $\kappa_\lambda \in \mathbb{N}_{++}$

$$\max_{(a,\varepsilon)\in \mathbb{G}_a^\lambda \times \mathbb{G}_\varepsilon} \left| \frac{\lambda_{i+1}(a,\varepsilon) - \lambda_i(a,\varepsilon)}{1+\lambda_i(a,\varepsilon)} \right| < 10^{-\kappa_\lambda}$$

   Otherwise go back to step 4 with $\lambda_i \to \lambda_{i+1}$.

Full Pseudo-Algorithm: Lastly, we must provide an algorithm for computing equilibrium prices. The following algorithm recognizes that the wage can be expressed in terms of the interest rate:

$$w(r) = (1-\alpha)K(r)^\alpha L^{-\alpha}$$

where

$$K(r) = \left(\frac{\alpha}{r+\delta}\right)^{\frac{1}{1-\alpha}} L$$

Therefore we will find a market clearing interest rate. We will use the bisection method to find an interest rate for which the excess demand function is zero, although there are several other rootfinding methods that one could use (you should attempt to experiment!).

1. Set initial boundaries on the interest rate:

$$[\underline{r}_0, \bar{r}_0] = \left[-\delta + 0.01,\ \frac{1}{\beta} - 1 - 0.01\right]$$

   and set the initial interest rate to

$$r_0 = \frac{1}{2}\left(\underline{r}_0 + \bar{r}_0\right)$$

2. Given r_i, compute the savings policy function $a'(a,\varepsilon|r_i)$

3. Given r_i, compute the stationary distribution function $\lambda(a,\varepsilon|r_i)$
   and corresponding density function $f(a,\varepsilon|r_i)$

4. Update the interest rate using bisection.
   - Compute excess demand:

$$E(r_i) = K(r_i) - A(r_i)$$

$$K(r_i) = \left(\frac{\alpha}{r_i+\delta}\right)^{\frac{1}{1-\alpha}} L$$

$$A(r_i) = \sum_{\varepsilon \in \mathbb{G}_\varepsilon} \sum_{a \in \mathbb{G}_a} a'(a, \varepsilon | r_i) f(a, \varepsilon | r_i)$$

- Compute $\hat{r} = \alpha A(r_i)^{\alpha - 1} L^{1-\alpha} - \delta$
- If $E(r_i) > 0$ then increase the lower bound on the interval: $[\underline{r}_{i+1}, \bar{r}_{i+1}] = [\hat{r}, \bar{r}_i]$
- If $E(r_i) < 0$ then decrease the upper bound on the interval: $[\underline{r}_{i+1}, \bar{r}_{i+1}] = [\underline{r}_i, \hat{r}]$
- Set $r_{i+1} = \frac{1}{2}(\underline{r}_{i+1} + \bar{r}_{i+1})$

5. If for fixed $\kappa \in \mathbb{N}_{++}$, $|r_{i+1} - r_i| < 10^{-\kappa}$ then stop;
   otherwise, go to step 2 with $r_i \to r_{i+1}$.

4.4 Precautionary Savings in General Equilibrium

Precautionary Savings: Define the level of aggregate capital accumulated for self-insurance as $K(r_{ss}) - K(r^*)$, where r_{ss} is the equilibrium interest rate with incomplete markets and $r^* = \beta^{-1} - 1$ is the interest rate under complete markets. This is a measure of the precautionary motive.

Aiyagari (1994) calibrates and computes the general equilibrium allocation and prices. It is instructive to consider two extreme cases. When $\sigma \to 1$ so that $u(c) = \log(c)$ and if shocks are purely transitory such that $\rho = 0$ and $\varepsilon_t = \exp(\epsilon_t)$, then the interest rate is approximately $r \approx 1/\beta - 1$. This is the interest rate that prevails when asset markets are complete, suggesting that there is no precautionary savings motive under this particular assumption on primitives. On the other hand, Aiyagari (1994) shows that if agents are very risk averse with $\sigma = 5$ and highly persistent income shocks $\rho = 0.9$, then precautionary savings are a large fraction of aggregate output, $(K(r_{ss}) - K(r^*))/Y_{ss} = 14\%$.

Intuitively, when shocks are persistent, then a low income realization can remain low for a long time. But when σ is high, consumption volatility has a large negative impact on utility, since agents highly value smooth consumption paths. When agents are not very risk averse (σ is relatively low) and shocks are not persistent, then agents are less sensitive to fluctuations in consumption. Furthermore, shocks are mean reverting and average out over time, so that agents do not need to accumulate a large buffer stock in order to self insure.

When parameters are chosen in a more empirically relevant range ($\sigma = 2$ and $\rho = 0.9$), precautionary savings constitutes 5% of aggregate output. Also note that the precautionary motive is partially disciplined by the market clearing mechanism. The interest rate clears markets but also interacts with individual incentives to self insure.

Comparative Statics: When $\underline{a} < 0$, thereby loosening the no borrowing constraint, individuals can better smooth consumption. As a result, we should expect precautionary savings to decrease. In fact, asset supply decreases in partial equilibrium (i.e. with a constant interest rate). In general equilibrium, the interest rate increases to encourage individuals to hold more assets, that are demanded by the representative firm.

On the other hand, if agents become more risk averse due to an increase in the coefficient of relative risk aversion, then we would expect an increase in precautionary savings. Again, asset supply will increase in partial equilibrium and the interest rate will decrease in general equilibrium. The lower interest rate encourages individuals to hold fewer assets so that there is no excess supply relative to the representative firm's demand.

Lastly, consider the effect of an increase in the variance of income shock innovations, σ_ε^2. Again, agents will increase their precautionary savings in order to smooth consumption against larger idiosyncratic risk. Because the unconditional variance of the income process is $\sigma_\varepsilon^2/(1-\rho^2)$, an increase in ρ has the same qualitative effect as an increase in σ_ε.

4.4.1 Government Debt

Aiyagari and McGrattan (1998) construct a model in which the government can issue public debt afforded by distortionary income taxation. Government debt loosens individual borrowing constraints by increasing the supply of assets, but also crowds out productive capital. However, there are competing effects of a lower wage (due to less productive capital) and higher returns to savings. In net, the authors find that the average level of post-war US debt ($\approx 2/3$ of GDP) is very close to the optimal level from the perspective of their model.

Production: A representative firm produces output according to a Cobb-Douglas production function, $Y = zF(K, N)$, taking capital K and labor N as inputs. The firm rents capital and labor services in perfectly competitive factor markets with given prices for capital $r + \delta$ and labor w. Therefore, the firm chooses capital and labor to equate marginal products with factor prices:

$$w_t = (1-\alpha)F_N(K_t, z_t N_t)$$

$$r_t = (\alpha F_K(K_t, z_t N_t) - \delta)$$

Assume that $z_t = (1+g_z)^t$, where g_z is the growth rate of labor-augmenting technology. For simplicity, assume that $g_z = 0$ throughout.

Consumer's Problem: There is a unit continuum of infinitely-lived consumers. Consumers are endowed with a unit of time each period, used for either leisure or labor activity. Consumers value consumption and leisure according to the utility function $u(c, l)$, satisfying all of the usual properties and specified as a Cobb-Douglas aggregator over consumption and leisure nested in a CRRA utility function. Consumers choose their consumption, savings and labor effort according to the following problem:

$$v(a, \varepsilon) = \max_{c, a', l} \frac{(c^\eta l^{1-\eta})^{1-\sigma}}{1-\sigma} + \beta \sum_{\varepsilon'} \pi(\varepsilon'|\varepsilon) v(a', \varepsilon')$$

$$\text{s.t.} \quad c + a' \leq \bar{w}\varepsilon(1-l) + (1+\bar{r})a + T$$
$$a' \geq \underline{a}$$

where $\underline{a}$ is the consumer's debt limit and T is a lump-sum governmental transfer. Additionally, individuals are taxed on their total income, $w\varepsilon(1-l) + ra$, at linear rate τ. Therefore, $\bar{w} \equiv (1-\tau)w$ and $\bar{r} \equiv (1-\tau)r$.

The intratemporal optimality condition is:

$$u_l(c, l) = \bar{w}\varepsilon u_c(c, l)$$

and the Euler equation is:

$$u_c(c, l) \geq \beta(1+\bar{r}) \sum_{\varepsilon'} \pi(\varepsilon'|\varepsilon) u_c(c', l')$$

which holds with equality when the borrowing constraint is slack. Note that:

$$u_c(c, l) = \eta \frac{\left(c^\eta l^{1-\eta}\right)^{1-\sigma}}{c}$$

$$u_l(c, l) = (1-\eta) \frac{\left(c^\eta l^{1-\eta}\right)^{1-\sigma}}{l}$$

which implies the intratemporal optimality condition (at an interior solution) is:

$$(c/l) = \frac{\eta}{1-\eta}(1-\tau)w\varepsilon$$

And therefore marginal utility of consumption is:

$$u_c(c,l) = \left[\eta\left(\frac{1-\eta}{\eta\bar{w}\varepsilon}\right)^{(1-\eta)(1-\sigma)}\right]c^{-\sigma}$$
$$\equiv \psi h(\varepsilon)c^{-\sigma}$$

So that the Euler equation can be rewritten as (at an interior solution for l):

$$h(\varepsilon)c^{-\sigma} \geq \beta(1+\bar{r})\sum_{\varepsilon'}\pi(\varepsilon'|\varepsilon)h(\varepsilon')(c')^{-\sigma}$$

Therefore the utility specification reduces to one that only depends on consumption (at an interior solution) and labor productivity shocks. The specification implies that agents' marginal utility consumption increases when agents are more productive (receive higher wages).

Government: The government consumes G each period. The government affords consumption by issuing an income tax on capital and labor income $\tau(wN + rA)$, issuing debt $B' - B$ to be repaid at rate r, and issuing a lump-sum transfer from households T. Therefore the government's budget constraint is:

$$G + T + rB = B' - B + \tau(wN + rA)$$

Market Clearing: Given a distribution over individual wealth and income, $\lambda(a,\varepsilon)$, labor demand by firms must equal the supply of labor:

$$N = \sum_{\varepsilon}\int_{\mathcal{A}}\varepsilon\Big(1 - l(a,\varepsilon)\Big)\lambda(da,\varepsilon)$$
$$A = \sum_{\varepsilon}\int_{\mathcal{A}}a\lambda(da,\varepsilon)$$

and the supply of savings by households must equal the demand for capital by firms and government debt:

$$A \equiv \sum_{\varepsilon} \int_{\mathcal{A}} a\lambda(da, \varepsilon)$$

$$A = K + B$$

Therefore, total assets A are given by both capital and government debt.

Lastly, we derive the resource constraint by first substituting the firm's FOCs into the governmental budget constraint at steady state (e.g. $B' = B$):

$$G + T + rB = \tau(wN + r(K + B))$$

$$G + T + rB = \tau(zF(K, N) - \delta K) + \tau r B$$

$$G + T = \tau(zF(K, N) - \delta K) - (1 - \tau)rB$$

and aggregating the individual budget constraints and substituting the firm's optimality conditions:

$$C + K = \bar{w}N + (1 + \bar{r})(K + B) + T$$

$$C = (1 - \tau)(wN + rK) + (1 + (1 - \tau)r)B + T$$

$$C = (1 - \tau)(zF(K, N) - \delta K) + (1 + (1 - \tau)r)B + T$$

Lastly, substituting the governmental budget constraint, the resource constraint is:

$$\boxed{C + K + G = zF(K, N) + (1 - \delta)K + B}$$

Calibration: Aiyagari and McGrattan assume that government expenditures equal a fraction of output $G = gY$, where their sample average gives $g \equiv G/Y = 0.217$. They also assume that government transfers are a fraction of output so that $\chi \equiv T/Y = 0.082$. Lastly, government debt is two-thirds of output during the sample period os that $b = B/Y = 0.67$. Rewrite the resource constraint as:

$$c + g = 1 - \delta k + b \quad \Longrightarrow \quad c + \delta k = 1 - g + b \approx 1.45$$

where $c = C/Y$ and $k = K/Y$.

Other parameters are more standard. Labor share is $1 - \alpha = 0.70$, CRRA is $\sigma = 1.5$ which is close to log-utility, $\beta \approx 0.99$ and depreciation is $\delta = 0.075$ which imply an interest rate of 4.5%. The borrowing constraint is assumed to be $\underline{a} = 0$.

The parameter η can be chosen so that the model's steady state conditions imply a elasticity of labor supply of 2%. The resulting estimate is $\eta = 0.328$.

The stochastic process is parameterized with $\rho = 0.6$ and $\sigma_\varepsilon = 0.3$ which implies an unconditional variance of $0.09/(1 - 0.06) = 0.14$. Note that this is a less persistent process than is used in more modern calibrations. Given both a low-end coefficient of relative risk aversion and a low persistence process, we would expect that agents in this economy have a relatively weak precautionary motive.

Lastly, if we allow for a positive growth rate for TFP, then set $\gamma_z = 0.0185$ corresponding to post-war average growth until the 1990s. The normalized, stationary consumer's problem then has an effective discount factor of $\beta(1 + g_z)^{\eta(1-\sigma)} = 0.988$ and the effective interest rate is $\bar{r} - g_z$.

Results: Aiyagari and McGrattan (1998) compute their model under the calibration we have just described. Then the authors compute the optimal level of debt in the economy, which is the level of debt that maximizes average utility across agents. They find that the optimal quantity of debt is almost equal to the average debt-to-GDP ratio observed in the US (until the 1990s). Therefore, their model and calibration suggests that the US debt system "exactly balances the negative role of crowding out private capital and distorting labor supply and savings decisions through higher taxes" with the positive role of liquidity provision.

4.4.2 Constrained Efficiency

We have shown that incomplete market economies deliver different allocations than complete market economies. In particular, the precautionary savings motive generates a larger aggregate capital stock than the complete market economy. Given that the complete market economy is Pareto efficient (e.g. the primitives of the economy are sufficient to ensure that the First Welfare Theorem holds), then the incomplete market economy yields a Pareto inefficient allocation.

While the incomplete market equilibrium allocation is Pareto inefficient relative to the Social

Planner's allocation, the Planner's allocation is achieved by completing asset markets and therefore by changing the market structure. However, what if the Planner is required to choose individual allocations that obey market restrictions such as market incompleteness and market clearing prices, as well as obey individual budget and borrowing constraints? The welfare maximizing allocation that satisfies these constraints is called a *constrained efficient allocation.*

Constrained Efficiency: Dávila, Hong, Krusell and Ríos-Rull (2012) study the welfare properties of the growth economy with incomplete markets. In particular, they study a planning problem in which the Planner faces the same constraints that individuals face in the decentralized equilibrium. The recursive formulation of the Social Planner's Problem is:

$$\Omega(\lambda) = \max_{c(a,\varepsilon),a'(a,\varepsilon)} \sum_{\varepsilon} \int_A u\Big(c(a,\varepsilon)\Big)\lambda(da,\varepsilon) + \beta\Omega(\lambda')$$

$$\begin{aligned} \text{s.t.} \quad c(a,\varepsilon) + a'(a,\varepsilon) &\leq w(K)\varepsilon + (1 + r(K))a \\ a'(a,\varepsilon) &\geq \underline{a} \\ \lambda'(\mathcal{A},\mathcal{E}) &= \sum_{\varepsilon\in E} \int_A \pi(\varepsilon' \in \mathcal{E}|\varepsilon)\mathbb{1}\left[a'(a,\varepsilon) \in \mathcal{A}\right]\lambda(da,\varepsilon) \\ K(\lambda) &= \sum_{\varepsilon\in E} \int_A a\lambda(da,\varepsilon) \\ L(\lambda) &= \sum_{\varepsilon\in E} \int_A \varepsilon\lambda(da,\varepsilon) \\ R(K) &= F_K(K,L) + (1-\delta) \\ W(K) &= F_L(K,L) \end{aligned}$$

Again, the Planner can allocate consumption and savings to individuals, but allocations must satisfy individual budget and borrowing constraints, as well as market clearing conditions.

Given a distribution λ and savings tomorrow a'', the Planner instructs an agent with state (a,ε) to save according to the policy function $g(a,\varepsilon)$ that maximizes:

$$\sum_{\varepsilon} \int \left[u\left(F_L(K)\varepsilon + F_K(K)a - g(a,\varepsilon)\right) + \beta\sum_{\varepsilon'} \pi(\varepsilon'|\varepsilon)u\left(F_L(K')\varepsilon' + F_K(K')g(a,\varepsilon) - a''\right)\right]\lambda(da,\varepsilon)$$

where

$$K' = \sum_{\varepsilon} \int g(a,\varepsilon)\lambda(da,\varepsilon)$$

For *any* variation $\delta_{(g,\hat{\varepsilon})}$ of the optimal policy function $g(a,\varepsilon)$ and for any small perturbation

$\epsilon \neq 0$ the policy rule

$$\hat{g}(a, \varepsilon) = g(a, \varepsilon) + \epsilon \delta_{(\hat{a}, \hat{\varepsilon})}(a, \varepsilon)$$

should be suboptimal. Therefore define:

$$\Psi(\epsilon) = \sum_{\varepsilon} \int_A \Big[u\left(F_L(K)\varepsilon + F_K(K)a - \hat{g}(a, \varepsilon)\right)$$

$$+ \beta \sum_{\varepsilon'} \pi(\varepsilon'|\varepsilon) u\left(F_L(K^\epsilon)\varepsilon' + F_K(K^\epsilon)\hat{g}(a, \varepsilon) - a''\right) \Big] \lambda(da, \varepsilon)$$

where

$$\hat{K}' = \sum_{\varepsilon} \int_A \hat{g}(a, \varepsilon) \lambda(da, \varepsilon)$$

and take the derivative with respect to ϵ and evaluate at $\epsilon = 0$:

$$\Psi'(0) = \sum_{\varepsilon} \int_A \Big[-u'\left(F_L(K)\varepsilon + F_K(K)a - g(a, \varepsilon)\right) \delta_{(\hat{a}, \hat{\varepsilon})}(a, \varepsilon)$$

$$+ \beta \sum_{\varepsilon'} \pi(\varepsilon'|\varepsilon) u'\left(F_L(K')\varepsilon' + F_K(K')g(a, \varepsilon) - a''\right) F_K(K') \delta_{(\hat{a}, \hat{\varepsilon})}(a, \varepsilon) \Big] \lambda(da, \varepsilon)$$

$$+ \sum_{\varepsilon} \int_A \Big[\beta \sum_{\varepsilon'} \pi(\varepsilon'|\varepsilon) u'\left(F_L(K')\varepsilon' + F_K(K')g(a, \varepsilon) - a''\right)$$

$$\cdot \left(F_{LK}(K')\varepsilon' + F_{KK}(K')g(a, \varepsilon)\right) \sum_{\tilde{\varepsilon}} \int_A \delta_{(\hat{a}, \hat{\varepsilon})}(\tilde{a}, \tilde{\varepsilon}) \lambda(d\tilde{a}, \tilde{\varepsilon}) \Big] \lambda(da, \varepsilon)$$

Since the suboptimality holds for any variation, I choose a particular one:

$$\delta_{\hat{a}, \hat{\varepsilon}}(a, \varepsilon) = \mathbb{1}[\varepsilon = \hat{\varepsilon}] \cdot \mathbb{1}[a \geq \hat{a}]$$

Therefore:

$$\Psi'(0) = \int_{\hat{a}}^{\bar{a}} \Big[-u'\left(F_L(K)\hat{\varepsilon} + F_K(K)a - g(a, \hat{\varepsilon})\right)$$

$$+ \beta \sum_{\varepsilon'} \pi(\varepsilon'|\hat{\varepsilon}) u'\left(F_L(K')\varepsilon' + F_K(K')g(a, \hat{\varepsilon}) - a''\right) F_K(K') \Big] \lambda(da, \hat{\varepsilon})$$

$$+ \sum_{\varepsilon} \int_A \Big[\beta \sum_{\varepsilon'} \pi(\varepsilon'|\varepsilon) u'\left(F_L(K')\varepsilon' + F_K(K')g(a, \varepsilon) - a''\right)$$

$$\cdot \left(F_{LK}(K')\varepsilon' + F_{KK}(K')g(a, \varepsilon)\right) \int_{\hat{a}}^{\bar{a}} \lambda(da, \hat{\varepsilon}) \Big] \lambda(da, \varepsilon)$$

Since $\Psi'(0) = 0$ for any $\hat{a}$, then $\partial\Psi'(0)/\partial\hat{a} = 0$ as well:

$$\frac{\partial\Psi'(0)}{\partial\hat{a}} = \Big[- u'\left(F_L(K)\hat{\varepsilon} + F_K(K)\hat{a} - g(\hat{a},\hat{\varepsilon})\right)$$

$$+ \beta \sum_{\varepsilon'} \pi(\varepsilon'|\hat{\varepsilon}) u'\left(F_L(K')\varepsilon' + F_K(K')g(\hat{a},\hat{\varepsilon}) - a''\right) F_K(K')$$

$$+ \sum_{\varepsilon} \int_A \beta \sum_{\varepsilon'} \pi(\varepsilon'|\varepsilon) u'\left(F_L(K')\varepsilon' + F_K(K')g(a,\varepsilon) - a''\right)$$

$$\cdot \left(F_{LK}(K')\varepsilon' + F_{KK}(K')g(a,\varepsilon)\right) \lambda(da,\varepsilon)\Big] \lambda(d\hat{a},\hat{\varepsilon})$$

Lastly, since $\partial\Psi'(0)/\partial\hat{a} = 0$ for any $(\hat{a},\hat{\varepsilon})$ we obtain the Planner's Euler equation:

$$u_c(a,\varepsilon) \geq \beta R(K) \sum_{\varepsilon'} \pi(\varepsilon'|\varepsilon) u_c(a'(a,\varepsilon),\varepsilon') + \beta \sum_{\varepsilon'} \int_A u_c(a',\varepsilon') \left(F_{KK}a' + F_{LK}\varepsilon'\right) \lambda(da',\varepsilon')$$

where for notational compactness we define:

$$u_c(a,\varepsilon) \equiv u'\left(W(K)\varepsilon + R(K)a - a'(a,\varepsilon)\right)$$

Note that the individual agent would optimize using the following Euler equation:

$$u_c(a,\varepsilon) \geq \beta R(K) \sum_{\varepsilon'} \pi(\varepsilon'|\varepsilon) u_c(a'(a,\varepsilon),\varepsilon')$$

The Planner chooses an allocation that accounts for the effect of individuals on prices, which effectively means that the planner takes derivatives with respect to prices while individual agents do not and can not. Put differently, a *pecuniary externality* operates in this economy (agents do not internalize the effect their savings decisions have on prices) that the Planner corrects by redistributing resources across individuals.

This point can be seen by taking the difference between the individual's Euler equation and the Planner's Euler equation when evaluating at the same point in state space (a,ε) and with the same policy function a':

$$\Delta = \beta \sum_{\varepsilon'} \int_A u_c(a',\varepsilon') \left(F_{KK}a' + F_{LK}\varepsilon'\right) \lambda(da',\varepsilon') \tag{20}$$

This term captures the effect of an additional unit of savings on total income. Increased savings increases aggregate capital, which in turn increases the wage and lowers the interest rate. Furthermore, an increase in aggregate capital increases agents' exposure to income risk through a higher wage, $W\varepsilon$, while lowering agents' risk-free component of total income, which is given by the lower interest on savings Ra.

If $\Delta = 0$ then the competitive equilibrium allocation is constrained efficient, otherwise if $\Delta \neq 0$ then the competitive equilibrium allocation is constrained inefficient. Furthermore, the empirically relevant sign for Δ should be positive, implying that the Planner increases savings relative to the competitive equilibrium allocation! Mechanically, this is because low wealth agents receive a majority of their total income in risky labor income $W\varepsilon$, while high wealth agents receive a majority of their total income in deterministic capital income Ra. Because of the concavity utility function, low wealth agents receive a higher weight in Δ. Therefore the planner wishes to increase aggregate capital to boost the income of the lower wealth through a higher wage.

Euler's Theorem: Furthermore, if $F(K, L)$ is homogeneous of degree one (as is a Cobb-Douglas production function) then we can use Euler's Theorem (Proposition 1) to prove the following lemma:

Proposition 3
Let $F : \mathbb{R}^n_+ \to \mathbb{R}$ be twice continuously differentiable on $\mathbb{R}^n_{++}$. If F is homogeneous of degree k then $\partial F(x)/\partial x_j$ is homogeneous of degree $k-1$ for any $j = 1, \dots, n$.

Proof. Recall from Euler's Theorem:

$$kF(x) = \sum_{i=1}^{n} \frac{\partial F(x)}{\partial x_i} \cdot x_i$$

Differentiating each side with respect to x_j:

$$k\frac{\partial F(x)}{\partial x_j} = \sum_{i=1}^{n} \frac{\partial}{\partial x_j}\left(\frac{\partial F(x)}{\partial x_i}\right) + \frac{\partial F(x)}{\partial x_j}$$

$$(k-1)\frac{\partial F(x)}{\partial x_j} = \sum_{i=1}^{n} \frac{\partial^2 F(x)}{\partial x_i \partial x_j}$$

Therefore the partial derivatives of F are homogeneous of degree $k-1$. □

A simple corollary to Proposition 3 is that if the production function $F(K, L)$ is homogeneous of degree 1 then:

$$F_{KK}(K, L)K + F_{KL}(K, L)L = 0$$

Substitute this into result into equation (20) to obtain:

$$\Delta = \beta F_{KK} K \sum_{\varepsilon'} \int_A u_c(a', \varepsilon') \left(\frac{a'}{K} - \frac{\varepsilon'}{L} \right) \lambda(da', \varepsilon')$$

While we know that $F_{KK}(K, L) < 0$ by assumption on F, we must know the sign of the integral term to know the sign of Δ. In order to understand this difference in competitive and centralized Euler equations, it is instructive to study several special cases.

Special Cases: First, consider an economy with a representative agent and only two states for $\varepsilon \in \{\underline{\varepsilon}, \bar{\varepsilon}\}$ with $\underline{\varepsilon} < \bar{\varepsilon}$ and probabilities $\{\pi(\varepsilon)\}_{\varepsilon \in E}$. Then $a' = K$, $\lambda(a', \varepsilon') = \pi(\varepsilon')$ and:

$$\begin{aligned}
\Delta &= \beta F_{KK} K \sum_{\varepsilon'} u_c(a', \varepsilon') \left(1 - \frac{\varepsilon'}{L} \right) \pi(\varepsilon') \\
&= \beta F_{KK} K u_c(a', \bar{\varepsilon}) \left[\pi(\underline{\varepsilon}) \frac{u_c(a', \underline{\varepsilon})}{u_c(a', \bar{\varepsilon})} \left(1 - \frac{\underline{\varepsilon}}{L} \right) + (1 - \pi(\underline{\varepsilon})) \left(1 - \frac{\bar{\varepsilon}}{L} \right) \right] \\
&= \beta F_{KK} K u_c(a', \bar{\varepsilon}) \left[\pi(\underline{\varepsilon}) \frac{u_c(a', \underline{\varepsilon})}{u_c(a', \bar{\varepsilon})} \left(1 - \frac{\underline{\varepsilon}}{L} \right) + (1 - \pi(\underline{\varepsilon})) \left(1 - \frac{\bar{\varepsilon}}{L} \right) + \pi(\underline{\varepsilon}) \left(1 - \frac{\underline{\varepsilon}}{L} \right) - \pi(\underline{\varepsilon}) \left(1 - \frac{\underline{\varepsilon}}{L} \right) \right] \\
&= \beta F_{KK} K u_c(a', \bar{\varepsilon}) \left[\pi(\underline{\varepsilon}) \left(\frac{u_c(a', \underline{\varepsilon})}{u_c(a', \bar{\varepsilon})} - 1 \right) \left(1 - \frac{\underline{\varepsilon}}{L} \right) + 1 - \left((1 - \pi(\underline{\varepsilon})) \frac{\bar{\varepsilon}}{L} + \pi(\underline{\varepsilon}) \frac{\underline{\varepsilon}}{L} \right) \right] \\
&= \beta \underbrace{F_{KK}}_{<0} K u_c(a', \bar{\varepsilon}) \cdot \pi(\underline{\varepsilon}) \underbrace{\left(\frac{u_c(a', \underline{\varepsilon})}{u_c(a', \bar{\varepsilon})} - 1 \right)}_{>0} \underbrace{\left(1 - \frac{\underline{\varepsilon}}{L} \right)}_{>0} \\
&< 0
\end{aligned}$$

Therefore, a representative agent's allocation will be constrained inefficient. Furthermore, the result implies that if the Planner were to engineer a decrease in capital K, then welfare would improve

Suppose now that there are two types of agents with savings $\underline{a}' < \bar{a}'$ and mass $\lambda(\underline{a}')$ and $\lambda(\bar{a}')$. This can be distribution over a' thought of as arising from the initial distribution over wealth. Then we can write aggregate capital as $K = \sum_{a'} a' \lambda(a')$:

$$\begin{aligned}
\Delta &= \beta F_{KK} K \sum_{a'} \lambda(a') \sum_{\varepsilon'} \pi(\varepsilon') u_c(a', \varepsilon') \left(\frac{a'}{K} - \frac{\varepsilon'}{L} \right) \\
&= \beta F_{KK} K \sum_{a'} \lambda(a') u_c(a', \bar{\varepsilon}) \cdot \left[\pi(\underline{\varepsilon}) \left(\frac{u_c(a', \underline{\varepsilon})}{u_c(a', \bar{\varepsilon})} - 1 \right) \left(\frac{a'}{K} - \frac{\underline{\varepsilon}}{L} \right) + \frac{a'}{K} - \left((1 - \pi(\underline{\varepsilon})) \frac{\bar{\varepsilon}}{L} + \pi(\underline{\varepsilon}) \frac{\underline{\varepsilon}}{L} \right) \right]
\end{aligned}$$

$$= \sum_{a'} \lambda(a') \cdot \beta F_{KK} K u_c(a', \bar{\varepsilon}) \left[\pi(\underline{\varepsilon}) \left(\frac{u_c(a', \underline{\varepsilon})}{u_c(a', \bar{\varepsilon})} - 1 \right) \left(\frac{a'}{K} - \frac{\underline{\varepsilon}}{L} \right) + \left(\frac{a'}{K} - 1 \right) \right]$$

Accordingly, define:

$$\Delta(a') \equiv \pi(\underline{\varepsilon}) \left(\frac{u_c(a', \underline{\varepsilon})}{u_c(a', \bar{\varepsilon})} - 1 \right) \left(\frac{a'}{K} - \frac{\underline{\varepsilon}}{L} \right) + \left(\frac{a'}{K} - 1 \right)$$

We can show that $\Delta(a') < 0$ if and only if $a' > K$. Simply, if $a' > K$ then $a'/K > 1 > \underline{\varepsilon}/L$. But then the square bracketed term is positive while $F_{KK} < 0$. But then $\Delta(a') > 0$ when a' is sufficiently small.

The overall effect $\Delta = \sum_{a'} \lambda(a')\Delta(a')$ is ambiguous however. It is instructive to consider one more special case, one without uncertainty over income shocks. Let $\underline{\varepsilon} = \bar{\varepsilon} = L$ and rewrite:

$$\Delta = \beta F_{KK} K \sum_{a'} \lambda(a') \; u_c(a', L) \left(\frac{a'}{K} - 1 \right)$$

Holding $\bar{a}'$ fixed, when $\underline{a}'$ becomes sufficiently small then $\Delta < 0$. If the utility function satisfies Inada conditions and convex marginal utility, then the marginal utilities for low savings types will be larger than high savings types. Furthermore, low savings types have $a'/K < 1$ while high savings types have $a'/K > 1$. Therefore, if $\lambda(a')$ is a uniform distribution then as long as $\underline{a}' < \bar{a}$ we will have $\Delta < 0$. In order to have $\Delta > 0$ there must be a sufficiently smaller mass of low savings types than high savings types.

Quantitative Analysis: The authors calibrate the economy in a standard way. We discussed the calibration in Aiyagari (1994) in the last section. The authors find that the constrained efficient level of aggregate capital is 3.65 times the competitive equilibrium level. This suggests that the pecuniary externality is positive and very large.

5 Incomplete Markets with Aggregate Uncertainty

In the last section we studied stationary equilibria. In a stochastic steady state, individual agents reproduce aggregate outcomes each period and aggregate variables are time invariant. One can think of a stationary equilibrium as a long-run outcome of an economy with heterogeneity and uncertainty at the individual level, but no uncertainty at the aggregate level.

However, for many questions in macroeconomics, aggregate uncertainty is at the core of the issue. For example, in order to study business cycles there must be some notion of aggregate risk usually modeled as an aggregate TFP shock. When the economy contains aggregate shocks, aggregate outcomes such as aggregate savings or the joint distribution of wealth and income are now time-varying.

In this section we will extend the growth model with incomplete markets and idiosyncratic shocks of Section 4 to include aggregate productivity shocks. We will then discuss computational issues that arise. Lastly we calibrate the model and show that there is a tight connection between the marginal propensity to consume and the effect of heterogeneity on aggregate outcomes.

5.1 Recursive Competitive Equilibrium

In this section I describe the economic environment and define an equilibrium. There is no proof of existence for this particular equilibrium and therefore I leave issues of existence and uniqueness aside.[10]

The economy consists of a continuum of individual agents and a representative firm. Time is discrete and infinite.

Production: There is a representative firm that operates a constant returns to scale production technology, taking aggregate capital and labor as inputs at given factor prices (w, r). Let the production technology take the Cobb-Douglas form $zK^{\alpha}L^{1-\alpha}$, for $\alpha \in (0, 1)$, where z is a productivity shock.

Uncertainty: There are two types of shocks, idiosyncratic labor productivity shocks $\varepsilon \in \{\underline{\varepsilon}, \bar{\varepsilon}\}$ and aggregate productivity shocks $z \in \{\underline{z}, \bar{z}\}$ with $\underline{\varepsilon} < \bar{\varepsilon}$ and $\underline{z} < \bar{z}$. Therefore the stochastic state of the economy is (ε, z). Given the two-state process over idiosyncratic labor productivity, we will refer to these as employment shocks. In the low state $\underline{\varepsilon} = 0$ and in the high state $\bar{\varepsilon} = 1$. For aggregate productivity shocks, we will refer to the low state a bust or recession and refer to the high state as a boom or recovery.

Stochastic state variables evolve according to a four-state Markov chain, in which the dis-

[10] Miao (2006) proves the existence of a sequential equilibrium of an environment that generalizes the Krusell and Smith (1998) economy. However, Miao (2006) cannot prove that his sequential formulation yields Krusell and Smith's (1998) wealth-recursive representation.

tribution of idiosyncratic shocks depends on the aggregate shock, and the aggregate shock evolves according to a autoregressive process. The conditional probability that the aggregate shock z' is realized next period given this period's shock is z is denoted $\pi_z(z'|z)$. The joint conditional probability that an agent realizes individual shock ε' and aggregate shock z' next period given today's individual and aggregate shocks are (ε, z) is denoted $\pi(\varepsilon', z'|\varepsilon, z)$. Lastly, let the mass agents with shock ε when the aggregate state is z be given by $\pi_\varepsilon(\varepsilon|z)$.

Assume that the stochastic processes for aggregate and idiosyncratic shocks satisfy the following conditions:

$$\sum_{z'}\sum_{\varepsilon'} \pi(\varepsilon', z'|\varepsilon, z) = 1 \quad \forall\ \varepsilon, z$$

$$\sum_{\varepsilon'} \pi(\varepsilon', z'|\varepsilon, z) = \pi_z(z, z') \quad \forall\ \varepsilon, z, z'$$

The first assumption requires that π is a density function over (ε', z') for each (ε, z). The second assumption is a result of the law of large numbers: absent individual risk, the Markov chain reduces to the aggregate process for uncertainty.

Also suppose the processes satisfy the following assumptions on the correlation between aggregate and idiosyncratic risk:

$$\pi_\varepsilon(\underline{\varepsilon}|\bar{z}) < \pi_\varepsilon(\underline{\varepsilon}|\underline{z})$$

$$\pi(\bar{\varepsilon}, \underline{z}|\underline{\varepsilon}, \underline{z}) < \pi(\bar{\varepsilon}, \bar{z}|\underline{\varepsilon}, \underline{z})$$

$$\pi(\bar{\varepsilon}, \underline{z}|\bar{\varepsilon}, \bar{z}) < \pi(\bar{\varepsilon}, \bar{z}|\bar{\varepsilon}, \bar{z})$$

The first condition says that employment shocks are more likely when productivity is high. The second condition requires that if the agent is unemployed and the economy is in a recession, then getting employment shock $(\bar{\varepsilon})$ is more likely if the economy gets a high aggregate shock (enters a recovery). Likewise, the third condition requires that if an agent is employed during a boom, then the agent is more likely to stay employed if the economy remains in a boom than if it enters a bust.

State Variables: As in the stationary economy in Section 4, the individual state is wealth and labor productivity shock (a, ε). It is clear that the agent needs to know these two variables in order to know their total income.

Now there are also two aggregate states, the aggregate productivity shock and the distribution of wealth and labor productivity shocks across agents, (λ, z). These two states are needed to compute current and future prices (wages and rental rates). To see why first note that aggregate labor is a given function of z since labor is supplied inelastically by agents and $\underline{\varepsilon} = 0$. Therefore, aggregate labor is equal to the measure of agents receiving an employment shock, $L(z) = \pi_\varepsilon(\bar{\varepsilon}|z)$. Next, notice that the firm chooses capital and labor and optimally sets marginal products equal to factor prices. Accordingly, the rental rate $r(K, z)$ and the wage rate $w(K, z)$ can be written as functions of capital and productivity:

$$w(K, z) = (1 - \alpha) z \Big(K/L(z)\Big)^{\alpha}$$

$$r(K, z) = \alpha z \Big(K/L(z)\Big)^{\alpha - 1} - \delta$$

where $\delta < 1$ is the depreciation rate of aggregate capital.

Also note that the capital stock is determined in equilibrium according to the market clearing condition that aggregate savings equals the demand for capital by the representative firm:

$$K(\lambda) = \sum_{\varepsilon} \int_A a \lambda(a, \varepsilon) da$$

and therefore aggregate capital depends on the distribution over wealth and labor productivity, λ. Therefore, in order to forecast future prices, the agent needs to forecast next period's capital stock. But in order to forecast the future capital stock, agents must forecast the entire distribution over wealth and labor productivity.

Denote the law of motion of the distribution as $\lambda' = H(\lambda, z, z')$. The law of motion is constructed similarly to the stationary case. Given a savings policy function $g(a, \varepsilon)$ we can construct the transition function P such that:

$$\lambda'(\mathcal{A} \times \mathcal{E}) = \int_{A \times E} P\left((a, \varepsilon), \mathcal{A} \times \mathcal{E} | z, z'\right) d\lambda \tag{21}$$

where

$$P\left((a, \varepsilon), \mathcal{A} \times \mathcal{E} | z, z'\right) = \sum_{\varepsilon' \in \mathcal{E}} \pi(\varepsilon', z' | \varepsilon, z) \mathbb{1}[g(a, \varepsilon) \in \mathcal{A}] \tag{22}$$

Given this law of motion for the distribution, the agent can compute next period's capital as:

$$K(\lambda') = \sum_{\varepsilon'} \int_A a' \lambda'(a', \varepsilon') da'$$

Agent's Problem: Individual agents have preferences over consumption described by

$u(c) = c^{1-\sigma}/(1-\sigma)$, for $\sigma > 1$. In the recursive problem, the individual agent's state vector is $(a, \varepsilon; \lambda, z)$, as previously discussed. The recursive formulation of the agent's problem is:

$$v(a, \varepsilon; \lambda, z) = \max_{c,a'} u(c) + \beta \sum_{\varepsilon'} \sum_{z'} \pi(\varepsilon', z' | \varepsilon, z) v(a', \varepsilon'; \lambda', z')$$

$$\begin{aligned} \text{s.t.} \quad c + a' &\leq w(K(\lambda), z)\varepsilon + \Big(1 + r(K(\lambda), z)\Big)a \\ a' &\geq \underline{a} \\ \lambda' &= H(\lambda, z, z') \end{aligned}$$

Recursive Competitive Equilibrium: Now define a recursive competitive equilibrium of this economy as a value function v, a savings policy function g and consumption policy function c, policies for the firm K and L, a law of motion for the distribution H, and prices w and r such that:

(i) Given (w, r, G), (c, g) solve the individual agent's problem with value function v.

(ii) Given (w, r), firms choose (K, L) optimally by setting marginal products to factor prices:

$$w(K, z) = (1-\alpha)\frac{zK^{\alpha}L(z)^{1-\alpha}}{L(z)}$$

$$r(K, z) = \alpha\frac{zK^{\alpha}L(z)^{1-\alpha}}{K} - \delta$$

(iii) Asset and labor markets clear:

$$K(\lambda) = \sum_{\varepsilon} \int_{\mathcal{A}} a \; d\lambda$$

$$L(z) = \sum_{\varepsilon} \int_{\mathcal{A}} \varepsilon \; d\lambda$$

and the goods market clears (which trivially holds by Walras' Law):

$$\sum_{\varepsilon} \int c(a, \varepsilon) d\lambda + K(\lambda') = zK(\lambda)^{\alpha} L(z)^{1-\alpha} + (1-\delta)K(\lambda)$$

(iv) and; law of motion H is generated by policy function g and Markov transition function π according to equations (21) and (22).

5.2 Computation and Near-Aggregation

Bounded Rationality: Because a functional - the distribution λ - is an element of the state vector, the state space is infinite dimensional. Carrying an infinite dimensional state renders this problem computationally intractable.

In order to circumvent the "curse of dimensionality," we will follow Krusell and Smith (1998) and others in reducing the size of the state space by replacing the distribution in the state vector with a finite list of moments and statistics from the distribution. We will consider the new state vector $(a, \varepsilon; K, z)$.

Implicit in this definition of the state vector is the assumption that the first moment of the distribution, K, is sufficient to characterize the distribution. This assumption is made to simplify explication of the method, but could easily be relaxed by adding additional moments to the state vector. In fact, Krusell and Smith (1998) find that replacing the distribution with the first moment, aggregate capital K, provides a good approximation insofar as the solution is very accurate.

We can justify this reduction in the size of the state space from first principles. If agents have partial information about the distribution, either due to bounded rationality or costs of obtaining information (see Young (2010)), then the evolution of the state of the economy must be consistent with the behavior of individuals who do not observe the entire distribution λ.

However, agents still need to forecast future prices. Without the distribution in the state vector, agents must know the law of motion for aggregate capital in order to forecast future prices. Again following Krusell and Smith (1998), we will guess a functional form for the law of motion for aggregate capital and verify that the function form is consistent with equilibrium when agents have partial information about the distribution. Specify the law of motion as the function $G(K, z)$:

$$K' = G(K, z) \equiv \beta_0(z) + \beta_1(z)K$$

where the coefficients are allowed to depend on aggregate productivity shocks.

The individual agent's problem then becomes:

$$v(a, \varepsilon; K, z) = \max_{c, a'} u(c) + \beta \sum_{\varepsilon'} \sum_{z'} \pi(\varepsilon', z' | \varepsilon, z) v(a', \varepsilon'; K', z')$$

$$
\begin{aligned}
\text{s.t.} \quad c + a' &\leq w(K,z)\varepsilon + \Big(1 + r(K,z)\Big)a \\
a' &\geq \underline{a} \\
K' &= \beta_0(z) + \beta_1(z)K
\end{aligned}
$$

Computation: I will discuss Krusell and Smith's (1998) method for computing the equilibrium allocation of this economy. The computational algorithm shares many of the same features as those in Section 4.3. However, there are three main differences. First, solving the individual's problem requires taking the law of motion for aggregate capital as given and using two new aggregate state variables K and z. This is not a substantial difference however. The second difference is that we can no longer solve for a steady state distribution, because in the presence of aggregate shocks the distribution is time varying.[11] The last difference is that we are no longer computing equilibrium by finding a steady state price, but instead are searching for an equilibrium law of motion for capital, which maps to an equilibrium evolution of prices.

Intuitively, Krusell and Smith (1998) resolve these differences by: (i) guessing a law of motion for capital and using it to compute the savings policy function, (ii) using the policy function, simulating a long panel of distributions and (iii) estimating the implied law of motion for capital from the simulated panel data to update the guess on the law of motion. An equilibrium outcome requires that the law of motion for capital implied by simulated data be the same as the one that agents used to forecast prices. Put differently, the law of motion that agents use to make decisions must be consistent with the aggregate outcomes generated by these agents.

The computational algorithm requires grids over assets, capital and the idiosyncratic and aggregate shocks so that the individual state space is $\mathbb{G}_a \times \mathbb{G}_\varepsilon \times$ and the aggregate state space is $\mathbb{G}_K \times \mathbb{G}_z$. It turns out that the capital grid does not require a large number of nodes or non-uniform spacing in order to generate accurate solutions to this model.

The following computational algorithm can be attributed to Young (2010):

1. Choose a number of periods T and draw[12] $\{z_t\}_{t=0}^{T}$ from the distribution $\pi(z_t|z_{t-1})$.

[11] There are alternative methods that approximate the distribution as a function of its moments. That is, for each (K,z) the method would solve for a different $\lambda(a,\varepsilon;K,z)$. This approach can substantially reduce the computational time costs. The interested reader should refer to Den Haan (1997), Algan et al. (2010) and Reiter (2010). In Sager (2015), I have developed a method that draws from this set these techniques.

[12] There are overhead costs to drawing random numbers and therefore you can reduce computational time a surprising amount by drawing all random numbers in advance of computation.

2. Guess a policy function for savings,

$$a_0'(a, \varepsilon; K, z) \text{ for all } (a, \varepsilon; K, z) \in \mathbb{G}_a \times \mathbb{G}_\varepsilon \times \mathbb{G}_K \times \mathbb{G}_z$$

and guess coefficients for the law of motion for capital $\{\bar{\gamma}_0(z), \gamma_0(z)\}_{z \in \mathbb{G}_z}$

3. Given the law of motion for aggregate capital at iteration i

$$G_i(K, z) = \bar{\gamma}_(i)(z) + \gamma_i(z)K$$

and given prices $w(K, z)$ and $r(K, z)$, compute updated policy functions $c_{i+1}(a, \varepsilon; K, z)$ and $a_{i+1}'(a, \varepsilon; K, z)$. For each $(a, \varepsilon; K, z)$, the policy functions satisfy the Euler equation, budget constraint and borrowing constraint:

$$u'\left(c_{i+1}(a, \varepsilon; K, z)\right) \geq \beta \sum_{\varepsilon', z'} \pi(\varepsilon', z'|\varepsilon, z) u'\left(c\Big(a_{i+1}'(a, \varepsilon; K, z), \varepsilon'; K', z'\Big)\right)(1 + r(K', z'))$$

$$c_{i+1}(a, \varepsilon; K, z) = w(K, z)\varepsilon + \Big(1 + r(K, z)\Big)a - a_{i+1}'(a, \varepsilon; K, z)$$

$$a_{i+1}'(a, \varepsilon; K, z) \geq \underline{a}$$

where $K' = G_i(K, z)$ for notational compactness. To compute policy functions, use one of the methods discussed in Section 4.3 (e.g. Fixed Point Iteration or Endogenous Gridpoint Method).

4. Construct a larger asset grid $\mathbb{G}_a^\lambda$ that is uniformly spaced and has a number of nodes given by $n_a^\lambda > n_a$. Set $\lambda_0(a, \varepsilon) = \lambda_{ss}(a, \varepsilon)$ for all $(a, \varepsilon) \in \mathbb{G}_a^\lambda \times \mathbb{G}_\varepsilon$ where λ_{ss} is the steady state distribution. Given $\lambda_t(a, \varepsilon)$, compute aggregate capital according to:

$$K_t = \sum_{\varepsilon \in \mathbb{G}_\varepsilon} \sum_{a \in \mathbb{G}_a^\lambda} a \lambda_t(a, \varepsilon)$$

and update the distribution according to:

$$\lambda_{t+1}(a', \varepsilon') = \sum_{\varepsilon \in \mathbb{G}_\varepsilon} \pi(\varepsilon', z_{t+1}|\varepsilon, z_t) \sum_{a \in \mathbb{G}_a^\lambda} \omega_j(a, \varepsilon; K_t, z) \lambda_t(a, \varepsilon)$$

for each $j = 1, \ldots, n_a^\lambda$, where weights for each $j = 1, \ldots, n_a^\lambda$ and $(a, \varepsilon) \in \mathbb{G}_a^\lambda \times \mathbb{G}_\varepsilon$ are given by:

- For all $j = 2, \ldots, n_a^\lambda - 1$:

$$\omega_j(a, \varepsilon; K_t, z_t) = \left\{ \begin{array}{ll} 1 - \dfrac{a'(a, \varepsilon; K_t, z_t) - a_j}{a_{j+1} - a_j} & \text{if } a_{j-1} \leq a'(a, \varepsilon; K_t, z_t) \leq a_j \\ \dfrac{a'(a, \varepsilon; K_t, z_t) - a_j}{a_{j+1} - a_j} & \text{if } a_j \leq a'(a, \varepsilon; K_t, z_t) \leq a_{j+1} \\ 0 & \text{otherwise} \end{array} \right\}$$

- For $j = 1$:

$$\omega_1(a, \varepsilon; K_t, z_t) = \left\{ \begin{array}{ll} 1 - \dfrac{a'(a, \varepsilon; K_t, z_t) - a_1}{a_2 - a_1} & \text{if } a_1 \leq a'(a, \varepsilon; K_t, z_t) \leq a_2 \\ 1 & \text{if } a'(a, \varepsilon; K_t, z_t) < a_1 \\ 0 & \text{otherwise} \end{array} \right\}$$

- For $j = n_a^\lambda$:

$$\omega_{n_a^\lambda}(a, \varepsilon; K_t, z_t) = \left\{ \begin{array}{ll} 1 & \text{if } a'(a, \varepsilon; K_t, z_t) > a_{n_a^\lambda} \\ \dfrac{a'(a, \varepsilon; K_t, z_t) - a_{n_a^\lambda - 1}}{a_{n_a^\lambda} - a_{n_a^\lambda - 1}} & \text{if } a_{n_a^\lambda - 1} \leq a'(a, \varepsilon; K_t, z_t) \leq a_{n_a^\lambda} \\ 0 & \text{otherwise} \end{array} \right\}$$

and where the policy function is evaluated at K_t according to interpolation along the K-dimension:

$$a'(a, \varepsilon; K_t, z_t) = \mathcal{I}\Big(K \in \mathbb{G}_K,\ a'(a, \varepsilon; K, z_t) \mid K_t\Big)$$

5. Given $\{\lambda_t\}_{t=0}^T$ and $\{K_t\}_{t=0}^T$, discard the first τ periods and compute new coefficients by estimating the relationship

$$K_{t+1} = \hat{\bar{\gamma}}(z) + \hat{\gamma}(z) K_t$$

via regression for each z. In other words, estimate separately the relationship on the subset of the data for which $z_t = \bar{z}$ and for which $z_t = \underline{z}$. Finally, update the coefficients incrementally by dampening:

$$\bar{\gamma}_{i+1}(z) = (1 - \zeta)\bar{\gamma}_i(z) + \zeta\hat{\bar{\gamma}}(z)$$

$$\gamma_{i+1}(z) = (1-\zeta)\gamma_i(z) + \zeta\hat{\gamma}(z)$$

where $\zeta \in (0,1)$ is a dampening parameter.

6. If $\bar{\gamma}_{i+1}(z) = \bar{\gamma}_i(z)$ and $\gamma_{i+1}(z) = \gamma_i(z)$ up to the desired tolerance for each z, then stop. Otherwise go back to step 3 with $\bar{\gamma}_i(z) \to \bar{\gamma}_{i+1}(z)$ and $\gamma_i(z) \to \gamma_{i+1}(z)$.

Note that, even if the law of motion coefficients have converged, it is still necessary to check the fit of the functional specification for the law of motion. It may be the case that the simulated data does not support an approximately linear relationship between capital today and capital tomorrow (e.g. the R^2 of the regression may be low). Some experimentation with functional forms may be required. In general, since higher order polynomials can approximate any function, we can always find an appropriate functional form. The issue is the computational cost from adding additional state variables. One possibility is to incrementally add moments and check the R^2 after each computation. If adding another moment does not improve the R^2 and the R^2 is sufficiently high then stop. Krusell and Smith (1998) take this latter approach and find that the first moment is sufficient to explain over 99% of the variation in the simulated data under the linear law of motion.

Calibration: Krusell and Smith (1998) choose standard values for most parameters: $\beta = 0.99$, $\delta = 0.025$, $\sigma = 1$ and $\alpha = 0.36$. Aggregate shocks are set to a one percent deviation from steady state, $\underline{z} = 0.99$ and $\bar{z} = 1.01$. The probability of receiving the unemployment shock is set to $\pi_\varepsilon(\underline{\varepsilon}|\bar{z}) = 0.04$ and $\pi_\varepsilon(\underline{\varepsilon}|\underline{z}) = 0.1$, reflecting 4% unemployment during booms and 10% unemployment during busts. Furthermore, $\pi(\varepsilon', z'|\varepsilon, z)$ is chosen so that the average duration of booms and busts is eight quarters, the average duration of unemployment is 1.5 quarters in booms and 2.5 quarters in busts, and lastly:

$$\frac{\pi(\underline{\varepsilon}, \underline{z}|\underline{\varepsilon}, \bar{z})}{\pi_z(\underline{z}|\bar{z})} = 1.25\frac{\pi(\underline{\varepsilon}, \underline{z}|\underline{\varepsilon}, \underline{z})}{\pi_z(\underline{z}|\underline{z})}$$

$$\frac{\pi(\underline{\varepsilon}, \bar{z}|\underline{\varepsilon}, \underline{z})}{\pi_z(\bar{z}|\underline{z})} = 0.75\frac{\pi(\underline{\varepsilon}, \bar{z}|\underline{\varepsilon}, \bar{z})}{\pi_z(\bar{z}|\bar{z})}$$

which is a condition on the probability of remaining unemployed when transitioning from a boom to a bust (which is assumed larger than when staying in a bust), and on the probability of remaining unemployed when transitioning from a bust to a boom (which is assumed lower than when staying in a boom).

Near-Aggregation: In equilibrium, Krusell and Smith (1998) find that the linear law of motion for the first moment of the wealth distribution provides a tight fit to the model-simulated data. In particular, $R^2 = 0.999998$ when conditioning on either $\underline{z}$ or $\bar{z}$. Agents who perceive the law of motion for aggregate capital follows the estimated linear form will make extremely small forecasting errors: forecasts of prices 25 years ahead have maximum errors of less than 0.1 percent. Therefore, the evolution of aggregate variables does not depend on the entire distribution or even changes in the high-order moments of the distribution. Apparently, the evolution of aggregate variables only depends on, approximately, the evolution of the first moment of the distribution.

Perhaps surprisingly, that aggregate capital is a sufficient statistic for aggregate outcomes is the result one would expect from an economy with a representative agent or with complete markets. Accordingly, the result is called *near-aggregation*. Near-aggregation arises for three main reasons.

1. Savings policy function is approximately linear, except for low wealth agents who are close to the borrowing constraint. These low wealth agents have precautionary savings motives, which generate large curvature in the savings policy function, but only in this region of the state space.

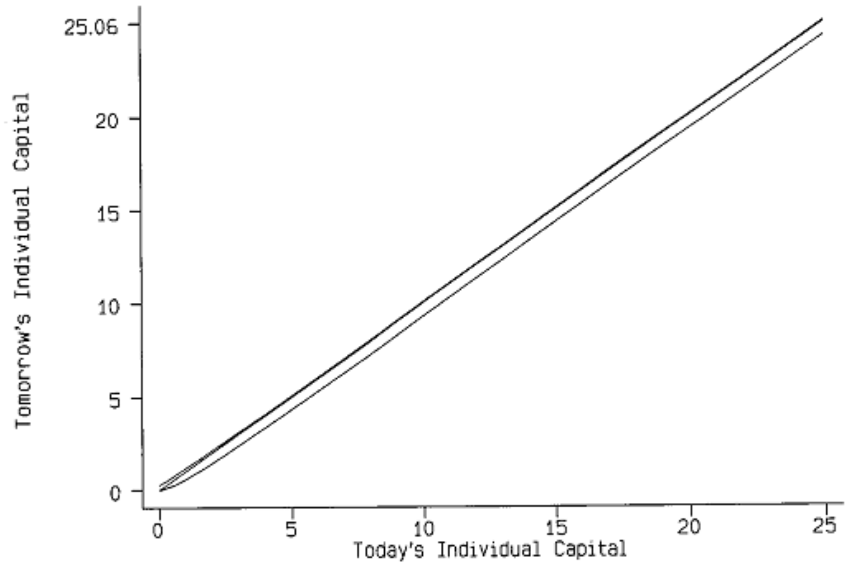
2. There is only a small mass of low wealth agents, whose savings policy functions exhibit large curvature. Most agents in this economy have a large buffer-stock of savings and largely self-insure against shocks. Therefore, most agents in this economy have high wealth and approximately linear savings policy functions.

3. Lastly, shocks are small and do not force agents to quickly draw down their savings. As a result, aggregate shocks do not force many agents into low wealth regions of the state space. The economy features a distribution that always has its most mass where the savings policy function is approximately linear.

Figure 1 shows that the savings policy function is in fact linear above a threshold for wealth. Therefore, most agents in this economy have the same marginal propensity to save out of wealth. In Section 1.2 we used the Gorman aggregation result to show that if all agents have the same marginal propensity to save out of wealth, then the economy admits a representative agent. In effect, if policy functions were of the form:

$$a'(a, \varepsilon) = \kappa_0(z) + \kappa_1(z)a + \kappa_2(z)\varepsilon$$

Figure 1: Savings policy function in Krusell and Smith (1998).



then aggregation would yield a linear law of motion:[13]

$$
\begin{aligned}
K' &= \sum_{\varepsilon} \int_A a'(a, \varepsilon; K, z)\lambda(da, \varepsilon) \\
&= \kappa_0(z) + \kappa_1(z)K + \kappa_2(z)L(z) \\
&\equiv \bar{\kappa}_0(z) + \kappa_1(z)K
\end{aligned}
$$

Therefore, because self-insurance is sufficient to smooth consumption, most agents in this economy have larger intertemporal incentives than precautionary ones. In this sense, the economy is close to a complete markets one.

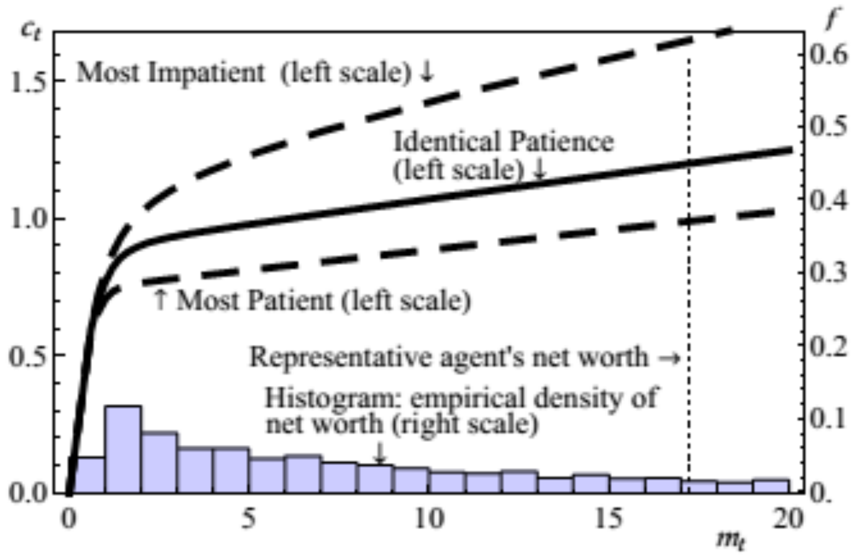
From a quantitative perspective, the result might not be entirely surprising. First off, Krusell and Smith (1998) choose $\sigma = 1$, which means that agents do not suffer high costs from fluctuations in utility (low risk aversion). Second, $\beta = 0.99$ is somewhat high and therefore agents do not have much incentive to borrow to consume (high patience) and prefer to save to intertemporally substitute consumption to the future. Lastly, aggregate shocks are small and agents can easily save without drawing down on their buffer stock of savings (little risk).

[13]This insight motivates Den Haan and Rendahl's (2010) Explicit Aggregation Method, a fast computational algorithm for heterogeneous agent economies with incomplete markets and aggregate uncertainty.

5.3 When Does Heterogeneity Matter?

Two recent papers – Carroll, Slacalek, Tokuoka and White (2014) and Krueger, Mittman and Perri (2015) – make an important point that Krusell and Smith's (1998) particular model and calibration imply a wealth distribution that is at odds with the empirically observed wealth distribution. The model neither generates enough concentration at the top of the wealth distribution, nor does it feature sufficiently many households at or close to zero wealth. These papers argue that the *near-aggregation* result is sensitive to the mass of agents at the bottom tail of the wealth distribution: without a sufficiently large mass of agents at the bottom tail of the wealth distribution, aggregate productivity shocks generate aggregate consumption responses that are essentially identical to those in the representative agent model.

Figure 2: Empirical Wealth Distribution and Consumption Policy Functions



These papers then study extensions of the basic model in Krusell and Smith (1998) that generate realistic wealth distributions. Both papers carefully re-specify the aggregate and idiosyncratic productivity shock processes, and allow for preference heterogeneity in discount factors. Carroll, Slacalek, Tokuoka and White (2014) demonstrate how the economy with an empirically relevant wealth distribution generates a much larger marginal propensity to consume at the aggregate level. Krueger, Mittman and Perri (2015) find that, similarly, a negative aggregate productivity shock generates a decline in aggregate consumption is substantially larger than in the representative agent economy. In both papers, larger consumption responses are primarily due to the existence of more wealth-poor households who respond strongly to aggregate shocks. The following Figure 2, taken from Carroll, Slacalek, Tokuoka and White (2014), illustrates that the consumption policy function has the highest curvature for low wealth (net worth) households, and therefore it will be these households that respond most strongly to changes in wealth.

Both of these papers have strong implications for policy. If the aggregate marginal propensity to consume is larger than standard models predict, and the measurement of the marginal propensity to consume depends crucially on wealth inequality, then government policies that redistribute wealth or help wealth-poor households insure against idiosyncratic and aggregate shocks can have very large aggregate welfare effects.

6 Consumer Finance and Insurance

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