Quality Ladders with Financial Frictions *

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PRELIMINARY AND INCOMPLETE

Abstract

This paper constructs a model of endogenous growth with quality ladders and capital markets imperfections. In particular, the model is a version of Kortum's (1997) model of technological change combined with Albuquerque and Hopenhayn's (2004) formulation of limited enforcement in firms' financial relationships. This is a theory of how weak investor protections generate less lending and less entrepreneurial financing. Since entrepreneurs innovate productivity-enhancing technologies, countries with larger financial frictions will exhibit lower levels of total factor productivity and output. Due to endogenous financial contracts, countries with larger financial frictions will also exhibit (*i*) a larger average firm size, (*ii*) a lower average firm-level productivity, and (*iii*) greater financial selection by intermediaries.

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1 Introduction

One of the central questions in economics is: Why are some countries rich while others are poor? There is enormous variation in per capita incomes across economies, and the gap between the richest and poorest countries has increased dramatically in the last century. To motivate the magnitudes, in 1988 output per worker in the United States was more than 35 times higher than output per worker in Niger. Furthermore, the proportional gap in GDP per capita between the richest and poorest countries grew more than five-fold from 1870 to 1990. The answer to this question is naturally important because it gives us insight into how governments and supranational entities can improve the welfare of the majority of the world's population.

Motivated by recent empirical evidence on the connection between entrepreneurship and financial contract enforcement, I articulate a theory of how weak investor protections generate less lending and therefore less entrepreneurial financing. Since entrepreneurs innovate productivity-enhancing technologies, less financially developed countries, e.g. in which financial markets unsuccessfully finance entrepreneurial innovations, will exhibit lower levels of total factor productivity and output. By focusing on the nexus of finance, entrepreneurship and innovation, my proposed theory stresses that financial markets are crucial to development.

Specifically, this paper constructs a model of endogenous growth with quality ladders and capital markets imperfections. The model is a version of Kortum's (1997) model of R&D combined with Albuquerque and Hopenhayn's (2004) formulation of limited enforcement in firms' financial relationships. By explicitly providing a role for financial frictions I obtain a structural model that can disentangle the channels by which technological adoption and financial frictions affect growth. I find that a preliminary version of the model is qualitatively consistent with the data: the model predicts that stronger contract enforcement generates more lending; and because less lending implies that fewer entrepreneurs receive financing, the model predicts a lower rate of entry and lower productivity.

In the model, less developed financial systems are in fact more selective with respect to financing new projects of varying productivity. This is because the enforcement friction lowers the rate at which financial intermediaries lend capital to firms, and therefore yields longer waiting periods until firms operate at their optimal scale. If a producer innovates a more productive technology before the older technology has reached its optimal scale, creditors will stop financing the old technology and shift resources to the new one. Because creditors will lose any sunk costs in the process, they will only finance sufficiently productive technologies – ones that are not at risk of being quickly made obsolete by innovations. On the other hand, incumbent firms will live longer because of the lower rate at which competing technologies obtain financing, thereby insulating them from competitive pressures. These effects suggest that pervasive limited enforcement impedes creative destruction.

The model also comments on a recent literature[1] that studies the affect on productivity of capital misallocation due to financial frictions. These studies construct models with heterogeneous firms, calibrate an exogenous productivity processes and then stationary equilibria.

[1] See Buera et al. (2011), Midrigan and Xu (2010) and Moll (2010).

Does entry and exit of firms along a growth path have different consequences for misallocation? Although I do not use the model along this dimension currently, future versions will be capable of measuring the extent of capital misallocation.

Background: A large empirical and theoretical literature has focused on determining the causes of global differences in economic development. The literature cites differences in capital stock, labor force size and quality, and the efficiency with which capital and labor services are utilized in production - also known as "total factor productivity." The consensus view is that total factor productivity accounts for approximately two-thirds of the cross-country differences in output per worker. The evidence further shows that productivity growth fueled high growth countries during the 20th century.[2] Consistent with the standard quality ladder model, I consider Joseph Schumpeter's theory of "Creative Destruction" and posit that entrepreneurs generate productivity growth by innovating new technologies.

If we want to understand why productivity differs across countries, it is instructive to examine systematic cross-country differences in the economic constraints that entrepreneurs face. Standard economic theory predicts that financial markets are the main mechanism by which businesses are matched with investors and receive financing for capital purchases. However, financial markets vary in the effectiveness with which they allocate funds - a claim that is supported by large cross-country differences in the volume of private credit as a fraction of GDP. Levine (1999) and Beck et al. (2000) substantiate a link between financial market development and productivity. They find a robust, positive correlation between financial development and both GDP growth and productivity growth. Furthermore, Rajan and Zingales (1998) provide evidence that causation runs from financial development to GDP and productivity growth.

Given the empirical evidence, I draw upon recent research that provides a structural rationale for why financial market development differs across economies. Djankov et al. (2007) and LaPorta et al. (1997) show that countries with legal systems that more effectively enforce contracts have better developed financial markets - as reflected by higher volumes of private credit claims as a fraction of GDP - than countries where contract enforcement is more lax. Financial markets undertake the activity of credit-creation by evaluating and underwriting lending contracts between investors and entrepreneurs. Therefore, since contractual arrangements form a basis of financial activities, legal systems that protect creditors and enforce contracts are more likely to finance the implementation of new technologies that increase productivity. Legal systems that provide lax contract enforcement will generate additional investment costs and ration credit to otherwise profitable businesses.

This paper is also related to a recent literature on the allocation of capital across firms. Hsieh and Klenow (2009) analyze cross-sectional firm-level data and find evidence of large misallocation in less developed financial markets. For example, they show that if capital and labor services were reallocated to the most productive firms, there would be a 50% productivity gain in China and a 60% productivity gain in India. Banerjee and Moll (2010) point out that the Indian and Chinese banking sectors are government-managed and are notoriously ineffective at contract enforcement. Furthermore, Beck et al. (2008) find that industries with a large fraction of small firms tend to grow faster in economies with more developed

[2]See Caselli (2005), Hall and Jones (1999), and Klenow and Rodriguez-Clare (1997).

financial markets. This suggests that financial development better allocates resources to smaller, high productivity firms that are otherwise credit constrained.

2 Model

Environment: Time is continuous. There are four types of agents in the economy: a representative consumer, a representative final good producer, a representative financial intermediary and a continuum of intermediate goods producers. There is no aggregate uncertainty. There is a unit mass of intermediate inputs denoted by $\omega \in [0,1]$ and a final consumption good.

Consumer's Problem: A representative household consumes final output $\{c_t\}_{t=0}^{\infty}$, owns productive capital $\{k_t\}_{t=0}^{\infty}$; receives income from rentals at given prices $\{r\}_{t=0}^{\infty}$; and trades claims on intermediate firms' equity value $\{b_t\}_{t=0}^{\infty}$ at given prices $\{q_t\}_{t=0}^{\infty}$. The household orders consumption streams according to a standard concave utility function $u(\cdot)$ and discounts the future at rate ρ. I assume that the capital stock does not depreciate. Lastly, the household owns intermediate firms and is rebated their aggregate profits, Π_t. Given initial capital and bond holdings, the household's problem is:

$$J_0(k_0, b_0) = \max_{\{c_t, k_t, a_t\}_{t=0}^{\infty}} \int_0^{\infty} e^{-\rho t} u(c_t) dt$$

$$\text{s.t.} \quad \dot{W}_t = r_t W_t + \Pi_t - c_t$$

where I define $W_t = k_t + q_t b_t$ as the household's wealth.

Final Good Producer: The final good producer transforms intermediate inputs $\{X_t(\omega)\}_{\omega \in [0,1]}$ into Y_t units of the consumption good according to a Cobb-Douglas production function. At each moment in time, the final good producer purchases inputs at given prices $\{p_t(\omega)\}_{\omega \in [0,1]}$.

$$\max_{\{X_t(\omega)\}_{t=0}^{\infty}} Y_t - \int_0^1 p_t(\omega) X_t(\omega) d\omega$$

$$\text{s.t.} \quad Y_t = \exp\left\{ \int_0^1 \ln\Big(X_t(\omega)\Big) d\omega \right\}$$

The final good sector is perfectly competitive and therefore the final good producer earns zero profits.

Intermediate Goods Production: In each sector $\omega \in [0,1]$ and at each instant, an intermediate good producer with factor productivity $A(\omega)$ transforms $K_t(\omega)$ units of capital into $A(\omega)K_t(\omega)$ units of the intermediate good. Production requires an initial fixed cost I and per period working capital costs $r_t K_t(\omega)$. Working capital is purchased in competitive capital markets at rental price r_t. Intermediate producers have zero initial net worth and therefore require external financing to pay the fixed cost of entry. Producers obtain financing by contracting with competitive financial intermediaries, which I discuss in the next section.

Intermediate producers engage in Bertrand competition in product markets. As is standard, the most productive firm charges the highest price at which it can undercut competitors.

The second-most productive firm earns zero profits at this price, while all other firms earn negative profits from production. Denote the first and second highest productivities in a sector as $\{A(\omega), A_{-1}(\omega)\}$, where $A(\omega) > A_{-1}(\omega)$. Then the price charged in this sector is $p(\omega) = r/A_{-1}(\omega)$.

Financial Intermediation: Intermediate producers seek external finances to fund their firm's initial fixed cost, I.[3] I follow Albuquerque and Hopenhayn (2004) in modeling a lending relationship in which competitive financial intermediaries offer ex-ante long-term contracts to intermediate producers. Intermediaries and firms are risk neutral and discount future payments according to the consumer's discount rate, q_t.

A *contract* is a list of functions $\mathcal{C} \equiv \{K, d, \tau\}$ that specifies the amount of loaned capital (K), the firm's payment (d) and the termination age for the contract (τ) as a function of the state vector $Z \equiv (t, z^t, A, A_{-1})$. The state vector consists of the firm's age (t), the history of stochastic events (z^t) which is described in the next section, and the first / second highest productivity levels in a given sector.

A contract is *feasible* if and only if it satisfies the following limited liability and enforcement constraints. The *limited liability constraint* ensures that the producer never pays the intermediary in excess of the project's revenues:

$$\text{(LL)} \qquad d_t \geq 0 \qquad \forall\, t$$

Second, I assume that at each t the producer can default on the contract and abscond with some fraction of the firm's capital stock. In this case the intermediary seizes the firm's equity value (V_t) but can only recover a fraction θ^{-1} of it. An imperfect recovery rate can be understood as firm liquidation and sale when the producer has private knowledge of how to operate the technology, as in Hart and Moore (1994). In order for contracts to be self-enforcing the payments to the firm must be high enough to discourage default, which yields the following *enforcement constraint*:[4]

$$\text{(EC)} \qquad V_t(A, A_{-1}) \equiv \mathbb{E}_z \int_t^{\tau} e^{-\int_0^s q_a da} d_s ds \geq \theta K_t \qquad \forall\, t$$

where $\int_0^t q_a da$ is the household's valuation of the firm. Hence $V_t(A, A_{-1})$ is defined as the expected NPV of payments to the producer as of time t.

The timing of the contract is as follows. At $t = 0$ competitive financial intermediaries offer contracts to an entrant firm. Suppose the entrant accepts a contract. At $t = 0$, the contract specifies that the intermediary funds the fixed cost and for all $t \leq \tau$, the intermediary advances working capital $r_t K_t$, which the producer can either use to purchase capital or can abscond with. If the producer defaults, the contract goes into arbitration over the residual value of the firm.[5] If the producer purchases capital he uses it to produce and then sells the

[3] In Albuquerque and Hopenhayn, the intermediary pays the fixed cost and loans the firm an initial amount of capital. That is true here as well, but with a qualification: The rental price is a flow variable, which is approximately zero on any small interval $[t, t + dt]$. Therefore, in continuous time, at the instant of entry the fixed cost is the the firm's only strictly positive cost.

[4] The value of θ can be determined by a more elaborate game between the lender and defaulting borrower. It is a reduced form parameter capturing bargaining parameters and liquidation value. I articulate the underlying bargaining problem in the final section of the paper.

[5] Contracts are self-enforcing and therefore there is no default along the equilibrium path.

output to the final good producer. Revenues are split between producer and intermediary according to the contract. All events are intraperiod, e.g. occurring within a given t. Finally, if $t = \tau$ then the intermediary terminates the contract before the transferring working capital to the producer.

A contract is *optimal under limited enforcement* if it is feasible and maximizes the firm's total surplus:[6]

$$S_0(A, A_{-1}) = \max_{\{K_t, d_t, \tau\}_0^\infty} \mathbb{E}_z \int_0^\tau e^{-\int_0^t q_s ds} \Big(p_t A K_t - r_t K_t \Big) dt$$
$$\text{s.t.} \quad \text{(LL) and (EC)}$$

Lastly, I assume that upon contract termination the firm's technology becomes public information whereby the producer is no longer capital constrained. He can directly rent capital from households and operate his technology at zero profit (due to limit pricing). He produces the residual quantity demanded after the most productive firm sells his output: $A_{-1}K_{-1} = (X - AK)$. I make this assumption because enforcement constraints effectively restrict production capacity. The assumption simplifies the distribution of capital over older technological vintages, which can become a large dimensional object in steady state.[7]

Stochastic Structure and Entry: In each sector $\omega \in [0, 1]$, a potential entrant draws new technological blueprints according to a Poisson process with arrival rate λ. Upon arrival, the blueprint's productivity is independently drawn from a Pareto distribution with tail index $\zeta > 1$: $F(z) = 1 - z^{-\zeta}$. Upon entry, producers receive no subsequent productivity draws.

When an intermediary offers a contract to an entrant, it faces the risk that another intermediary offers a contract to a more productive entrant in the future. Taking as given the decisions of intermediaries in the future, the intermediary only offers a contract today if its expected profits net of sunk cost break even: $S(A, A_{-1}) - V(A, A_{-1}) = I$. This means the intermediary only finances sufficiently productive entrants. Let $\bar{A}(A_{-1})$ be the set of entrant productivities to which the intermediary offers funding, given current incumbent A_{-1}. Then entry is a Poisson process with a probability that a blueprint arrives and is adopted in any time interval $[t, t + dt)$ given by:

$$Pr\Big(z \in \bar{A}(A_{-1}) \Big| A_{-1}\Big) = \lambda dt \int_{z \in \bar{A}(A_{-1})} dF(z) \equiv \lambda \beta(A_{-1}) dt$$

Equilibrium: An equilibrium of this economy is defined as an allocation and prices that satisfy:

[6]As shown in Albuquerque and Hopenhayn, optimal contracts are equivalent when (1) the intermediary maximizes his own stream of payments (D) or (2) the lender maximizes the total surplus of the project (S), subject to the same constraint set. This relies on the fact that total surplus is the sum of the producer and intermediary's payments: $S = V + D$. For ease of explication I choose the latter formulation.

[7]Without this assumption I would need to keep track of the distribution of capital over older technological vintages. If older vintage firms are capacity constrained then residual demand trickles down until demand is met. However, firms would choose different prices according to limit pricing. This additionally makes the market clearing condition complicated. I abstract from this more realistic vintage structure in exchange for this simplified representation of the relationship between innovation and vintages.

1. Given prices $\{q_t, r_t\}_{t=0}^{\infty}$ and initial allocation $\{b_0, k_0\}$, the allocation $\{c_t, k_t, b_t\}_{t=0}^{\infty}$ solves the consumer's utility maximization problem.

2. Given prices $\{\{p_t(\omega)\}_\omega\}_{t=0}^{\infty}$, the allocation $\{Y_t, \{X_t(\omega)\}_\omega\}_{t=0}^{\infty}$ solves the final good producer's profit maximization problem.

3. Given prices $\{q_t, r_t\}_{t=0}^{\infty}$, the contract $\mathcal{C} = \{K, d, \tau\}$ solves the intermediary's problem for each sector $\omega \in \Omega$.

4. Markets clear for final goods, $c_t + \dot{k}_t + I \int_\Omega \beta(\omega) d\omega = Y_t$, intermediate goods, $X_t(\omega) = \sum_i A_t^i(\omega) K_t^i(\omega)$, capital, $k_t = \int_\Omega \sum_i K_t^i(\omega) d\omega$, bonds $b_t = \int_\Omega [S_t(\omega) - V_t(\omega)] d(\omega)$ and intermediate firms' dividends are rebated to the household $\Pi_t = \int_\Omega d_t(\omega) d\omega$.

3 Stationary Equilibrium

In future work I will characterize a balanced growth path of this economy. In the current version of the paper, I will analyze a stationary partial equilibrium. In particular, I will impose that bond prices and rental prices (q, r) are stationary and final output demand is a constant, Y. Furthermore, for tractability, I will parameterize the fixed cost as $I = iY$.

3.1 Representative Consumer and Final Good Producer

The representative consumer's maximization problem is standard. Note that since households own intermediate firms, the household values the firms' future profits with discount factor $\exp(-qt)$.

Given prices and final demand Y, the final-good producer optimality requires $p(\omega)X(\omega) = Y$ for all $\omega \in [0, 1]$.

3.2 Optimal Contracts

Limited Enforcement Contract: Consider intermediate producers in market ω with first and second highest productivities (A, A_{-1}). The intermediary for A takes as given the adoption decision of future intermediaries, $\bar{A}(A)$, so that the pricing rule is:

$$p_t(z, A, A_{-1}) = \left\{ \begin{array}{ll} r/A_{-1} & \text{if } z \notin \bar{A}(A) \\ r/A & \text{if } z \in \bar{A}(A) \end{array} \right\}$$

Therefore when $z \in \bar{A}(A)$, the most productive firm becomes the second most productive firm and receives zero profits.

Recall that when $\theta > 0$, the enforcement constraint is $V_t \geq \theta K_t$. Differentiating the expression for V_t with respect to time we obtain a continuous time Bellman equation for the producer:

$$\delta V_t = d + \dot{V}_t$$

where I define the risk adjusted price of a share in the firm as: $\delta(A) \equiv q + \lambda\beta(A)$. Following Spear and Srivastava (1987), I include V as a state variable in the financial intermediary's problem and reformulate the contract recursively. Accordingly the above equation is the promise keeping constraint, which I now add to the contract space. Given state vector $Z = (t, z, A, A_{-1})$, the intermediary's recursive problem is:[8]

$$\delta S(Z, V) = \max_{K,d,\dot{V}} \Big(p(Z)AK - rK \Big) + S_v(Z, V)\dot{V}$$

$$\begin{aligned} \text{s.t.} \quad & \delta V = d + \dot{V} && \text{(PKC)} \\ & V \geq \theta K && \text{(EC)} \\ & d \geq 0 && \text{(LL)} \end{aligned}$$

This is essentially a continuous time analog of Albuquerque and Hopenhayn's principal-agent problem. As such their results hold in my environment, as shown in the next proposition.

However, my model has several key differences with Albuquerque and Hopenhayn's model: (a) continuous time instead of discrete time, (b) a linear production technology, (c) firms with heterogeneous productivity and (d) pricing determined by Bertrand competition. These features are sufficient to obtain closed form solutions for value functions, which Albuquerque and Hopenhayn do not obtain.

Claim 3.2.1 (Characterization)
Define $V^u \equiv \theta K^*$ as the smallest promise value for which the enforcement constraint does not bind. Then for all $V_t < V^u$ the optimal contract satisfies:

$$\begin{bmatrix} V_t \\ K_t \\ d_t \end{bmatrix} = \begin{bmatrix} 1 \\ \theta^{-1} \\ 0 \end{bmatrix} e^{\delta t} V_0$$

And for all $V_t \geq V^u$, the optimal contract satisfies:

$$\begin{bmatrix} V^* \\ K^* \\ d^* \end{bmatrix} \equiv \begin{bmatrix} V_t \\ K_t \\ d_t \end{bmatrix} = \begin{bmatrix} \theta \\ 1 \\ \delta\theta \end{bmatrix} \frac{X}{A}$$

The termination condition is: $\tau = \min\{t \mid z \in \bar{A}(A)\}$.

Proof. See Albuquerque and Hopenhayn (2004) and the appendix. □

The optimal contract backloads payments to the producer in two ways. First, the producer receives no payments until the firm receives the unconstrained capital advance. Second,

[8] I derive the Hamilton-Jacobi-Bellman equation in the appendix.

growth in the equity value is the highest feasible until the firm receives the unconstrained capital advance. The latter feature of the optimal contract occurs because the surplus value increases with the firm's equity value. Lastly, $V^* = V^u$ because delivering a higher equity value decreases the surplus value in this region of the state space.

Full Enforcement Contract: Under full enforcement, $(\theta = 0)$, the enforcement constraint never binds. Consequently the firm will operate at optimal scale until termination. Denote full enforcement objects with a superscript "f". Then $K^f = X/A$ for each t until termination. The termination condition, as before, allows the firm to operate until an entrant draws a sufficiently more productive technology, $\tau^f = \min\{t \mid z \in \bar{A}^f(A)\}$. The firm's expected total surplus is given by:

$$\delta S^f(Z) = (1 - \xi^{-1})Y^f$$

Note that when $I = 0$, the intermediary advances working capital (rK^f) each period and the producer repays out of its revenues. This environment is equivalent to the one in Kortum (1997).

3.2.1 Selection

Not all technologies receive financing. Because of the fixed cost of entry, the intermediary faces an opportunity cost of financing a project today. Namely, a more productive idea may arrive tomorrow after the intermediary has already sunk costs into a project. Therefore the intermediary only writes financial contracts with sufficiently productive firms. In this section I will derive a cutoff rule that characterizes the intermediary's decision.

Limited Enforcement: As shown in the previous section, the producer's stream of consumption must provide him with incentives to not default. The financial intermediary provides incentives by initiating the firm below full productive scale and puts off compensation to the producer until the firm has reached optimal scale. However, as I will show in this section, the rate at which the intermediary receives repayment is lower under limited enforcement. In order to offset the lower expected returns due to the entry of a more productive firm, intermediaries select more productive projects relative to an economy without enforcement frictions.[9]

Competitive intermediaries choose which entrants to offer contracts according to the following program:

$$\begin{aligned} D_0(Z_0) = \max_{x \in \{0,1\}} \quad & x \cdot \max_{V_0} \; S(Z_0, V_0) - V_0 - I \\ & \text{s.t.} \quad V_0 \geq V_0^*(A, A_{-1}) \end{aligned}$$

where $Z_0 \equiv (0, z^0, A, A_{-1})$ is the an entrant's state vector and V_0^* is the equilibrium value of the firms expected payments, which is determined by competition between intermediaries.

[9]This result is robust to the assumption of a representative intermediary. In a game between a continuum of competitive, risk neutral intermediaries, there exists a contract that allows any intermediary to exactly break even in expectations with his sunk cost. Offering a lower contract would not be individually rational, while perfect competition ensures that offering a contract with higher expected return could be undercut by a competitor.

As I show in the following proposition, two functions characterize the solution to the Intermediary's program: $V_0(A, A_{-1})$ and $\bar{A}(A_{-1})$. Respectively these are the expected payment to an entrant of type (A, A_{-1}) and the set $\bar{A}(A_{-1}) = \{A|x = 1, A_{-1}\}$ of entrant productivity levels for which the break-even constraint is satisfied.

Theorem 3.2.2 (Entry Cutoff)
For all $A_{-1} \in supp(F)$, there exist functions $A^*(A_{-1})$ and $V_0(A, A_{-1})$ such that

(i) If $Z_0 \equiv (0, z^0, A, A_{-1})$ then,

$$S\Big(Z_0, V_0(A, A_{-1})\Big) - V_0(A, A_{-1}) = I \qquad \forall\ A \geq A^*(A_{-1})$$

$$S\Big(Z_0, V_0(A, A_{-1})\Big) - V_0(A, A_{-1}) < I \qquad \forall\ A < A^*(A_{-1})$$

(ii) For all A_{-1}, $A^*(\cdot)$ solves

$$\delta\Big(A^*(A_{-1})\Big) \equiv q + \lambda A^*\Big(A^*(A_{-1})\Big)^{-\zeta} = \mathbb{W}\left(\frac{\theta}{i}\frac{A_{-1}}{A^*(A_{-1})}\right)\frac{\frac{A^*(A_{-1})}{A_{-1}} - 1}{\theta/r} \qquad \text{(FE)}$$

where $\mathbb{W}(\cdot)$ is *Lambert's W function.*

Proof. See appendix. □

Part (i) of the theorem states that the set $\bar{A}(A_{-1})$ follows a cutoff rule: given a current incumbent A_{-1}, the intermediary finances any entrant with productivity cutoff $A^*(A_{-1})$ or greater. A lower productivity level violates the intermediary's break-even constraint, leaving the intermediary with no incentive to offer a contract. Furthermore, the function $V_0(A, A_{-1})$ that solves the program is the unique function that satisfies both the break-even constraint with equality and the first-order condition, $S_v(Z_0, V) - 1 \leq 0$.

Part (ii) of the theorem characterizes the cutoff rule as the solution to a functional equation. The functional equation can be understood as describing a game played between current and future intermediaries. When a current intermediary chooses a cutoff rule, he must forecast the probability that a future intermediary finances an entrant. Since entrants become monopolists, future entry is equivalent to capital loss from the perspective of the current intermediary. Given the initial fixed cost of entry and the break even constraint, if a future intermediary is less selective (e.g. there is a high probability of capital loss) then the current intermediary will be more selective. Likewise, if future intermediaries are more selective then current intermediaries will be less selective. The solution to this game is constructed to be time consistent across generations of intermediaries. The functional equation embeds this tradeoff between current and future selection.

To further characterize the cutoff inventive step, I perform the following comparative static exercises.

Corollary 3.2.3 (Comparative Statics)
Define the vector $\kappa = (Y, q, \lambda, \zeta, \theta)$. Holding all remaining variables fixed,

$$\begin{bmatrix} D_q A^* > 0 \\ D_\theta A^* > 0 \\ D_i A^* > 0 \\ D_\lambda A^* > 0 \\ D_\zeta A^* < 0 \end{bmatrix}$$

Proof. See appendix. □

Each threshold has the same qualitative response to a parameter perturbation. The responses are fairly intuitive: (λ) a higher frequency of innovation increases opportunity costs of financing and therefore increases selection, (ζ) an increase in the Pareto tail index makes tail events less likely thereby decreasing the mean productivity draw and making intermediaries less selective, (q) more patient intermediaries are willing to wait longer to fund an entrant, and (i) from the break even constraint, higher startup costs clearly increase selection. Lastly, and importantly, higher enforcement frictions (θ) worsen incentive costs and increase selectivity.

Full Enforcement: Under full enforcement, a producer will participate in a financial contract as long as $V_t \geq 0$. Therefore the productivity cutoff of a future intermediary in a sector with productivity A_{-1} is defined by the break even constraint. Simple algebra yields:

$$\delta\Big(A^*(A_{-1})\Big) \equiv q + \lambda A^* \Big(A^*(A_{-1})\Big)^{-\zeta} = i\left(1 - \frac{A_{-1}}{A^*(A_{-1})}\right)$$

Notice that the LHS is decreasing in A^* and the RHS is increasing in A^*. Therefore, comparative statics with respect to the full enforcement cutoff rule are given by:

$$\begin{bmatrix} D_q A^* > 0 \\ D_\lambda A^* > 0 \\ D_\zeta A^* < 0 \\ D_i A^* < 0 \end{bmatrix}$$

Interestingly, a change in the fixed cost of entry (i) is associated with less selection. In the limited enforcement case, the opposite comparative static result holds.

Lastly, notice that the cutoff is bounded away from 1. In Kortum (1997), $i = 0$ and therefore $A^*(A_{-1}) = A_{-1}$ for all A_{-1}. This means that entrants produce whenever they have a marginal more productive technology. Thus the presence of a fixed cost increases selection relative to Kortum's environment.

3.3 Discussion

In order to get a better sense for how the selection cutoff varies with θ, I have computed the cutoff function $A^*(\cdot)$ with a high and low value of θ. These cutoffs are graphed in the upper panel of figure 1, in addition to the full enforcement cutoff rule. From the graph it is evident that larger θ corresponds to an increase in both the intercept and slope of the essentially linear cutoff rule.

In the lower panel I have plotted the distribution over productivity levels. The graph shows that the mass of productivity levels shift down when θ is high. Hence the financial selection mechanism decreases productivity in this economy.

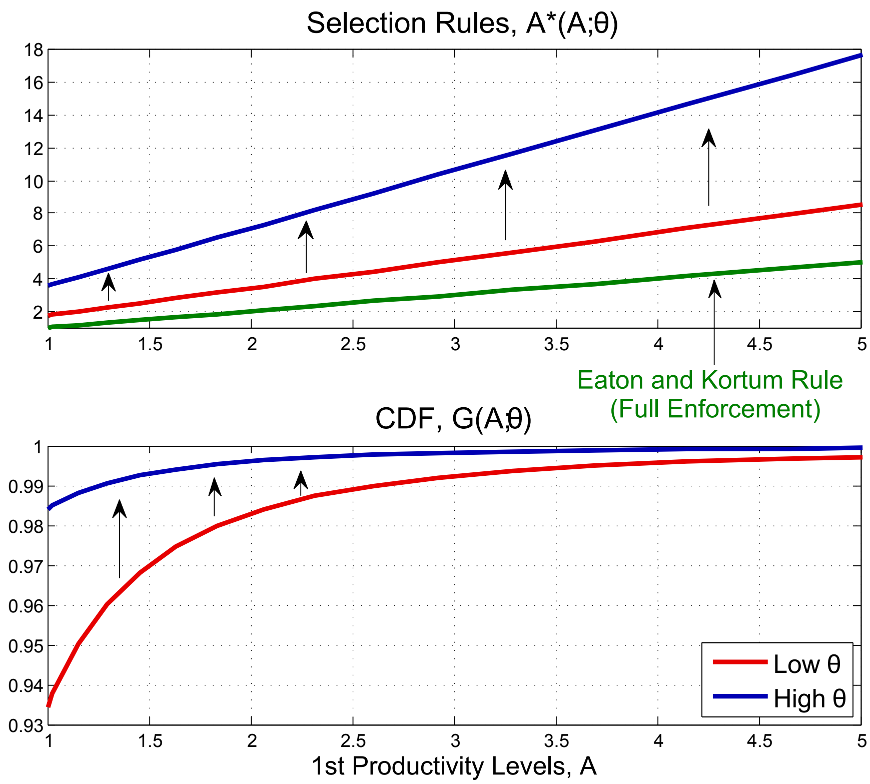

Figure 1:

These comparative statics are important for understanding firm dynamics within the model. Firm growth (δ), exit and entry $(Prob\{z \geq A^*\})$ are all governed by the cutoff rule. Since larger financial market imperfections increase selection, the model predicts that larger θ is associated with slower firm growth, low levels of churning, and a higher average firm size. Furthermore, the average firm will be less productive when θ is higher because increased financial selection thins out the right tail of the distribution.

Furthermore, increases in θ have a potentially large general equilibrium effect. In figure 2 I plot the prices (q, r) as a function of θ. I find that these prices are non-linearly related to θ. Because selection increases with q (e.g. $D_q A^* > 0$), the plot illustrates the general equilibrium effects that bolster selection. Potentially through these general equilibrium effects, under a more serious calibration, small changes in θ can lead to large changes in selection.

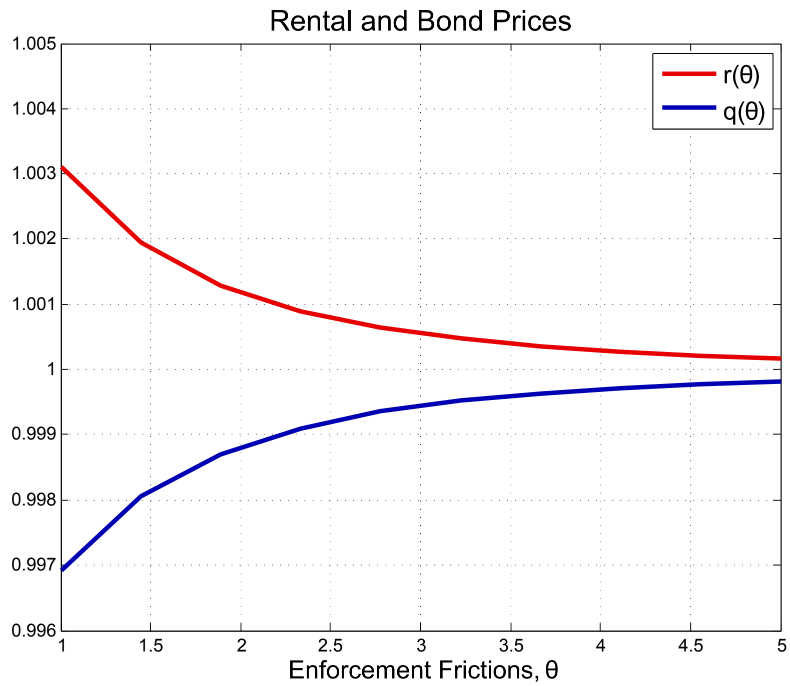

Figure 2:

4 Balanced Growth Equilibrium

[To Be Completed]

5 Calibration and Empirics

[To Be Completed]

Legal Institutions: Following Jermann and Quadrini (2009), entrepreneurs can default on loans and divert resources. Repayment then enters litigations and the two parties renegotiate the payment by bargaining over the joint surplus. As bargaining is intermediated by a country's juristic system, the bargaining weights $(\eta, 1-\eta)$ depend on the strength of a country's legal institutions. This idea is captured by a probability γ that the lender is awarded full remuneration while with probability $(1-\gamma)$ the lender is compensated a fraction $\chi \in (0,1)$ of the liquidated firm. Liquidation value is denoted by ψ. Therefore the bargaining problem is:

$$\max_{e} \Big(V - e\Big)^{1-\eta} \Big(e - \psi V\Big)^{\eta}$$

where $\eta = \gamma + (1-\gamma)\chi$. The solution is given by $e = (1 - \eta(1-\psi))V \equiv \theta^{-1}V$. Notice that countries with higher γ require higher remuneration to the plaintiff. The assumption is that countries with stronger legal systems can better identify and levy heavier punishments against those who breach contracts. Therefore lenders provide entrepreneurs with incentives to not engage in moral hazard and remain solvent by imposing the following enforcement constraint:

$$rK - e + V \leq V \qquad \Longrightarrow \qquad \theta r K \leq V$$

This enforcement constraint ensures that the entrepreneur has a larger NPV of repaying debt and continuing the project than diverting the loan. Higher η thus loosens the enforcement constraint and enables the entrepreneur to borrow more.

This game formalizes the notion that the efficacy of allocating capital to productive projects depends upon the strength of a country's legal institutions. This game will inform my future calibration of θ.

[To Be Completed]

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7 Mathematical Appendix

7.0.1 Derivation of the Frechet Distribution

Kortum (1997) uses search theoretic concepts to model technological innovation. I follow his derivation closely here.

Suppose researchers in country n expend labor effort at time t, L^R_{nt}, with productivity α_n. Suppose ideas are discovered according to a Poisson process. Then a new idea is discovered at a rate depending on the history of research effort: $\mu_{nt} = \alpha_n \int_{-\infty}^{t} L^R_{ns} ds$.[10] Therefore the number of ideas (N^i_t) that arrive in country n by time t is equal to a number k is:

$$Pr\{N^i_t = k\} = \frac{\mu^k_{nt}}{k!}\, e^{-\mu_{nt}}$$

When a productive idea is discovered it is uniformly allocated to a good $\omega \in [0, 1]$. The productivity (quality) of the new idea is drawn from a Pareto distribution with tail parameter θ: $F(Q) = Pr\{q \leq Q\} = 1 - Q^{-\theta}$ for $Q \geq 1$. Therefore, the probability that a new idea arrives at time t with productivity less than z is:

$$Pr\{Z \leq z\} = \begin{cases} \displaystyle\sum_{k=0}^{\infty} \frac{\left(\mu_{nt}\ (1 - z^{-\theta})\right)^k}{k!} \exp(-\mu_{nt}) & \text{for } z \geq \underline{z} = 1 \\ \exp(-\mu_{nt}) & \text{for } 0 \leq z \leq \underline{z} = 1 \\ 0 & o.w. \end{cases}$$

Note that $e^x = \sum_{k=0}^{\infty} x^k/k!$. Therefore z is a random variable that is drawn independently from a country-specific Frechet distribution, $G_n(z) = \exp(-\mu_{nt} z^{-\theta})$ for $z \geq 0$.

In my version of the derivation, I assume there is a single closed economy and that idea discovery does not require resources. Therefore in my model μ_{nt} is a constant, which I denote λ.

7.0.2 Derivation of HJB

As in the main text, the surplus value of the firm is given by:

$$S_0(A, A_{-1}) = \max_{\{K_t, d_t, \tau\}_0^\infty} \mathbb{E}_z \int_{t=0}^{\tau} e^{-qs} \Big(p_s(z_s, A_{-1}) A K_s - r_s K_s\Big) ds$$

$$\text{s.t. } V_t(A, A_{-1}) \equiv \mathbb{E}_z \int_t^{\tau} e^{-q(s-t)} d_s ds \geq \theta K_t \quad \forall\, t$$

$$d_t \geq 0 \quad \forall\, t$$

[10] Eaton and Kortum also consider the diffusion of technologies from other countries, which makes the arrival rate a weighted average of each country's μ_{nt}.

Given Bertrand competition in intermediate good markets, the price function is:

$$p_t(z, A_{-1}) = \begin{cases} r/A_{-1} & \text{if } z \notin \bar{A}(A) \\ r/A & \text{if } z \in \bar{A}(A) \end{cases}$$

Therefore, if $z \in \bar{A}(A)$ then the firm has zero flow of surplus.

Let $Z = (t, z, A, A_{-1})$ be a vector of state variables. Now we can write the Bellman equation as:

$$qS(Z,V) = \max_{\{K,d,\dot{V}\}} \Big(p(Z)A - r\Big)K + \frac{1}{dt}\mathbb{E}_z[dS]$$

$$\begin{aligned} \text{s.t. } \quad qV &= d + \tfrac{1}{dt}\mathbb{E}_z[dV] \\ V &\geq \theta K \\ d &\geq 0 \end{aligned}$$

where the Poisson process for z gives a probability $\lambda\beta dt$ on any interval of time $[t, t + dt]$, implying:

$$\frac{1}{dt}\mathbb{E}_z[dS] = \lambda\beta\Big(0 - S(Z, V_t)\Big) + \frac{1}{dt}(1 - \lambda\beta dt)\Big(S(Z, V_{t+dt}) - S(Z, V_t)\Big)$$

$$\frac{1}{dt}\mathbb{E}_z[dV] = \lambda\beta\Big(0 - V_t\Big) + \frac{1}{dt}(1 - \lambda\beta dt)\Big(V_{t+dt} - V_t\Big)$$

Substituting into the Bellman with $\delta = q + \lambda\beta$ and taking the limit as $dt \to 0$ we obtain:

$$\delta S(Z,V) = \max_{\{K,d,\dot{V}\}} \Big(p(Z)A - r\Big)K + S_v(Z,V)\dot{V}$$

$$\begin{aligned} \text{s.t. } \quad \delta V &= d + \dot{V} \\ V &\geq \theta K \\ d &\geq 0 \end{aligned}$$

7.0.3 Proof of Claim 3.2.1

Consider the Bellman function when $z = 0$. First notice that the objective function is strictly increasing in capital. Therefore the enforcement constraint binds when capital is below its unconstrained optimum, $K < K^*$. By applying standard dynamic programming methods, the value function is also increasing in V: $S_v(Z,V) \geq 0$.

Due to linearity, substitute the promise keeping and enforcement constraints into the objective function to obtain:

$$\delta S(Z,V) = \max_{d \geq 0} \Big(p(Z)A - r\Big)\frac{V}{\theta} + S_v(Z,V)(\delta V - d)$$

Clearly, when $S_v(Z,V) > 0$ the intermediary minimizes the payment to the producer: $d = 0$. Then the promise keeping constraint yields $\dot{V} = \delta V$, which is a linear differential equation with solution $V_t = e^{\delta t}V_0$. Therefore when the enforcement constraint binds, $K = \theta^{-1}V_t$.

The firm is unconstrained when it reaches its optimal level of capital, $K^* = X/A$ for all t. I will guess and verify that $\dot{V}^* = 0$, $V^* = \theta K^*$ and $d^* = \delta V^*$ for all t. When substituting the binding constrains into the Bellman, the resulting equation is a first order differential equation: $\delta V S_v(Z,V) - \delta S(Z,V) + \theta^{-1}(p(Z)A - r)V = 0$. For some constant c_0, the solution is:[11]

$$\delta S(Z,V) = \left[\delta c_0 - \frac{p(Z)A - r}{\theta}\log(V)\right]V$$

Under the proposed solution, once unconstrained, $V_t = V^*$ for all t. This gives rise to a boundary condition, $S_v(Z,V^*) = 0$. Using the boundary condition, solve for c_0 to obtain the solution:

$$S(Z,V) = \frac{p(Z)A - r}{\delta\theta}\left[1 + \log\left(\frac{V^*}{V}\right)\right]V$$

Therefore, $S_v(Z,V) < 0$ for $V > V^*$. Consistent with the guess, when $S_v(Z,V) < 0$ the optimal contract sets $\dot{V} = 0$ and $d = \delta V$. Finally, to verify the guess $V^* = \theta K^*$, suppose $V^* > \theta K^*$. When $V < V^u \equiv \theta K^*$, $S(Z,V) < S(Z,V^u)$. Hence, maximization requires minimizing V^*, which occurs at the binding enforcement constraint evaluated at K^*.

Another way to see why $V^* = V^u$, suppose to the contrary that $S_v(Z,V) > 0$ for all $V > V^u$. Then it is optimal to again set $d = 0$ and $\dot{V} = \delta V$. But then the producer's expected payments at entry are zero, which violates the sequential enforcement constraint. Therefore providing payment to the firm at the earliest feasible age loosens the enforcement constraint.

Lastly, characterize the termination condition by taking first order conditions with respect to τ.

$$\frac{1}{\theta}e^{-q\tau}\Big(p(Z)A - r\Big)V_\tau = 0$$

Therefore the FOC holds either if $V_\tau = 0$ or $p(Z)A = r$. The optimal contract rules out the former condition because $V_t = \min\{e^{\delta t}V_0, V^u\} > 0$ for all t. The latter condition is satisfied upon the arrival of a new entrant, $z \in \bar{A}(A)$. Therefore the termination condition is: $\tau = \min\{t \mid z \in \bar{A}(A)\}$.

7.0.4 Proof of Theorem 3.2.2

I will prove Theorem 3.2.2 in three steps. First I characterize the function V_0. Second I characterize x by proving that $\bar{A}(\cdot)$ follows a cutoff rule. Finally, I impose a consistency condition on $\bar{A}(\cdot)$ in order to characterize $\delta(\cdot)$. The analysis proceeds from characterizing the Intermediary's project choice decision, which is the solution to the following program:

$$D_0(Z_0) = \max_{x\in\{0,1\}} \quad x \cdot \max_{V_0} S(Z_0,V_0) - V_0 - I$$

[11] I derive this solution in the proof for claim 7.0.1.

$$\text{s.t.}\quad V_0 \geq V_0^*(A, A_{-1})$$

Immediately notice that when $x = 1$, competition among intermediaries drives expected profits to zero. Therefore the break-even constraint binds, $S(Z_0, V_0^*) - V_0^* = I$. Using the break-even constraint we can obtain a closed form solution for $V_0(\cdot,\cdot)$:

Claim 7.0.1 (Optimal Equity Value)
Given (A, A_{-1}), Y, and δ, the optimal equity value is:

$$V_0(A, A_{-1}) = \frac{\frac{\delta\theta}{\xi - 1} iY}{-\mathbb{W}\left(-\exp(-1)\cdot\frac{\delta i}{1-\xi^{-1}}\exp\left(\frac{\delta\theta}{\xi-1}\right)\right)} \equiv \frac{\delta\theta}{(\xi-1)}\cdot\frac{iY}{\phi(A, A_{-1})}$$

where $\mathbb{W}(\cdot)$ is the Lambert W function,[12] and $\xi \equiv A/A_{-1}$.

Proof. To prove Claim 7.0.1, I first state and prove a corollary to Claim 3.2.1. I show that we can characterize the age at which the firm becomes unconstrained.

Lemma 7.0.2 (Stopping Time)
There exists $T \in [0, \infty)$ such that $V_t < V^*$ for all $t < T$, and $V_t \geq V^*$ for all $t \geq T$. Such a T satisfies:

$$T = \frac{1}{\delta}\log(V^*/V_0)$$

Proof. From claim 3.2.1, if $V_t < V^*$ then $V_t = e^{\delta t}V_0$. Since $\delta(A) > 0$ for all A, V_t strictly increases with age. Then there exists some T for which:

$$T = \min_{\widehat{T}}\{\widehat{T} \mid e^{\delta\widehat{T}}V_0 \geq V^*\}$$

The set constraint holds with equality at the minimum of the set, which can be solved for T. □

In what follows it will be useful to note:

$$V^* = \theta K^* \quad \text{s.t. } K^* = X/A' = \xi^{-1}Y/r \quad \Longrightarrow V^* = \theta\xi^{-1}Y$$

To prove Claim 7.0.1, now derive total surplus:

$$S_0 = \int_0^T e^{-\delta t}(pA'K_t - rK_t)dt + \int_T^\infty e^{-\delta t}(pA'K^* - rK^*)dt$$

[12] This function behaves similarly to the log function. The log is the inverse function of $y = e^x$, while the Lambert function is the inverse of $y = xe^x$. Both functions equal 1 when evaluated at Euler's constant. However, Lambert's function equals zero when evaluated at zero and equals -1 when evaluated at $-\exp(-1)$. In fact Lambert's function has a domain $[-e^{-1}, \infty)$. See Corless, Gonnet, Hare, Jeffrey and Knuth (1996) for more details.

$$= \int_0^T e^{-\delta t}\left(1-\frac{1}{\xi}\right)\xi r \frac{e^{\delta t}V_0}{\theta}dt + \int_T^\infty e^{-\delta t}\left(1-\frac{1}{\xi}\right)pA'K^* dt$$

$$\delta S_0 = (\xi-1)\left(\frac{r}{\theta}T\delta V_0 + e^{-\delta T}\xi^{-1}Y\right)$$

Substituting the expression for T into the surplus function gives:[13]

$$\delta S(V_0) = (\xi-1)\left(\frac{r}{\theta}\frac{\log(V^*/V_0)}{\delta}\delta V_0 + \frac{V_0}{V^*}\xi^{-1}Y\right)$$

$$= \left(\frac{\xi-1}{\theta/r}\right)\left(\log(V^*) + \frac{\theta\xi^{-1}Y}{V^*}\right)V_0 - \left(\frac{\xi-1}{\theta/r}\right)\log(V_0)V_0$$

$$= \left(\frac{\xi-1}{\theta/r}\right)\left(\log(\theta\xi^{-1}Y)+1\right)V_0 - \left(\frac{\xi-1}{\theta/r}\right)\log(V_0)V_0$$

Competitive intermediaries offer the firm an equity value subject to a break even constraint $D_0 = S_0 - V_0 \geq I$. Perfect competition erodes away the intermediary's profits so that $V_0 \in \arg\max_V\{S(V) - V = I\}$. Therefore we can find a closed form solution for V_0 by using the intermediary's break even constraint.

$$\delta I = \delta S(V_0) - \delta V_0$$

$$\frac{\theta\delta I}{(\xi-1)r} = \left(\log\left(\frac{\theta Y}{\xi}\right)+1\right)V_0 - \log(V_0)V_0 - \frac{\theta\delta V_0}{(\xi-1)r}$$

$$\frac{\theta\delta I}{(\xi-1)r} = \left(\log\left(\frac{\theta Y}{\xi}\right)+1-\frac{\theta\delta}{(\xi-1)r}\right)V_0 - \log(V_0)V_0$$

It turns out that we can find a closed form solution for V_0 using the above expression. To do so we will make use of the Lambert function, $\mathbb{W}(\cdot)$.[14] To simplify the next few lines of derivation, let:

$$a = \frac{\theta\delta}{(\xi-1)r} \qquad \text{and} \qquad b = \log\left(\frac{\theta Y}{\xi}\right)+1$$

Then V_0 solves:

$$0 = aI - (b-a)V_0 + \log(V_0)V_0$$

$$V_0 = \exp\left(-\frac{aI}{V_0} - (a-b)\right)$$

$$-aI\exp\left(a-b\right) = -\frac{aI}{V_0}\exp\left(-\frac{aI}{V_0}\right)$$

$$-\frac{aI}{V_0} = \mathbb{W}\left(-aI\exp\left(a-b\right)\right)$$

[13]This is the same expression as we obtained from solving the Bellman equation in the previous section.
[14]This function behaves similar to the log function with a few differences. The log is the inverse function of $y = e^x$, while the Lambert function is the inverse of $y = xe^x$.

Or rewritten:

$$V_0 = \frac{-\theta\delta I}{(\xi-1)r}\,\mathbb{W}\left(\frac{-\theta\delta I}{(\xi-1)r}\exp\left(\frac{\theta\delta}{(\xi-1)r}-\log(V^*)-1\right)\right)^{-1}$$

$$= \frac{\frac{\theta\delta I}{(\xi-1)r}}{-\mathbb{W}\left(-\exp(-1)\cdot\frac{\delta i/r}{1-\xi^{-1}}\exp\left(\frac{\theta\delta}{(\xi-1)r}\right)\right)} \equiv \frac{\delta\theta}{(\xi-1)r}\cdot\frac{iY}{\phi(A,A_{-1})}$$

$\square$

Next I will characterize $\bar{A}(\cdot)$. First notice that $\bar{A}(\cdot)$ can be characterized by a cutoff rule. The objective is clearly increasing in A. The transition function, $\lambda\beta(A)$, is monotone since the tail measure decreases with A. Therefore, for any strictly increasing function $g : supp(F) \to \mathbb{R}$ and $A_1 > A_2$, the expected value is increasing in A:

$$\int g(A')Q(A,dA') = g(A_1)\cdot(1-\lambda\beta(A)) + g(A_2)\cdot\lambda\beta(A)$$

Therefore following Stokey, Lucas and Prescott (1989), Theorem 9.12, $S(\cdot,V)$ is strictly increasing in A. Then clearly a cutoff rule is optimal, denote it $A^*(A_{-1})$. Thus whenever $x=1$, $A \geq A^*(A_{-1})$ and the break-even constraint binds; and whenever $x=0$, $A < A^*(A_{-1})$ and $S - V < I$.

Next I will further characterize $A^*(A_{-1})$ as the solution to the functional equation (FE). Notice that when $x=1$, the FOC of the Intermediary's program with respect to V_0 is:

$$S_v(Z_0,V_0) = 1 \quad\Longleftrightarrow\quad \left(\frac{(\xi-1)r}{\delta\theta}\right)\log\left(\frac{V^*}{V_0}\right) = 1$$

Substituting this FOC into the break-even constraint, $S(Z_0,V_0) - V_0 = I$, yields:

$$\left(\frac{(\xi-1)r}{\delta\theta}-1\right)V_0 + \underbrace{\left(\frac{(\xi-1)r}{\delta\theta}\right)\log\left(\frac{V^*}{V_0}\right)}_{=1}V_0 = I \Longrightarrow V_0 = \left(\frac{\delta\theta}{(\xi-1)r}\right)I$$

which implies that $\phi(A,A_{-1}) = 1$ at the lowest (A,A_{-1}) for which the break-even constraint is satisfied.

Fix δ. Because the $\mathbb{W}(x)$ function is defined on a domain $x \in [-e^{-1},\infty)$, the function $V_0(\cdot,A_{-1})$ is defined for any A that satisfies:

$$-\exp(-1)\frac{\delta i/r}{1-\frac{A_{-1}}{A}}\exp\left(\frac{\delta\theta/r}{\frac{A}{A_{-1}}-1}\right) \geq -\exp(-1)$$

Therefore $\phi(A^*(A_{-1}),A_{-1};\delta) = 1$ when the condition holds with equality. To construct (FE), I will impose the above condition holds with equality and rewrite it as:

$$\frac{1-\frac{A_{-1}}{A^*(A_{-1})}}{\delta i/r} = \exp\left(\frac{\delta\theta/r}{\frac{A^*(A_{-1})}{A_{-1}}-1}\right)$$

$$\frac{\theta}{i} \cdot \frac{A_{-1}}{A^*(A_{-1})} = \frac{\delta\theta/r}{\frac{A^*(A_{-1})}{A_{-1}} - 1} \exp\left(\frac{\delta\theta/r}{\frac{A^*(A_{-1})}{A_{-1}} - 1}\right)$$

$$\mathbb{W}\left(\frac{\theta}{i} \cdot \frac{A_{-1}}{A^*(A_{-1})}\right) = \frac{\delta\theta/r}{\frac{A^*(A_{-1})}{A_{-1}} - 1}$$

Lastly, I took δ as given in the previous derivation. However, we previously defined $\delta \equiv \delta(A) \equiv q + \lambda A^*(A)^{-\zeta}$. This means that δ must satisfy a consistency condition: if $A^*(A_{-1})$ is the cutoff associated with A_{-1} then a future intermediary would choose a cutoff $A^*(A^*(A_{-1}))$, and subsequent intermediaries would also choose a cutoff of $A^*(\cdot)$ that depends on the productivity of current firm. Therefore by imposing this consistency condition, we obtain the functional equation, (FE):

$$\delta\Big(A^*(A_{-1})\Big) \equiv q + \lambda A^*\Big(A^*(A_{-1})\Big)^{-\zeta} = \mathbb{W}\left(\frac{\theta}{i}\frac{A_{-1}}{A^*(A_{-1})}\right)\frac{\frac{A^*(A_{-1})}{A_{-1}} - 1}{\theta/r}$$

which holds for all $A_{-1} \in supp(F)$.

7.0.5 Proof for Corollary 3.2.3

First I will prove an intermediate Lemma.

Lemma 7.0.3
If $A^*(\cdot)$ is strictly increasing for all A, then $\frac{d}{dA}\left(\frac{A^*(A)}{A}\right) \leq 0$.

Proof. By contradiction. Suppose $\frac{d}{dA}\left(\frac{A^*(A)}{A}\right) > 0$ on $[A_1, A_2] \subset [1, \infty)$. Suppose (FE) holds for all A. Take some $A \in (A_1, A_2)$ and $A' = A + \varepsilon$, for $\varepsilon > 0$ small. Since for all $x \in supp(F)$,

$$\mathbb{W}\left(\frac{\theta}{i} \cdot \frac{1}{x}\right) < \frac{x - 1}{\theta} r$$

then the RHS of (FE) increases as $A \to A'$. However since $A^*(\cdot)$ is strictly increasing, the LHS is decreasing as $A \to A'$. Therefore (FE) is violated and $\frac{d}{dA}\left(\frac{A^*(A)}{A}\right) > 0$ is a contradiction. □

Standard dynamic programming techniques, see Stokey, Lucas and Prescott (1989), confirm that the cutoff is strictly increasing for all A ($D_A A^*(A) > 0$).

To obtain the remaining comparative static results, rewrite the cutoff condition as:

$$\frac{1 - \frac{A}{A^*(A)}}{q + \lambda A^*(A^*(A))^{-\zeta}} = i \cdot \exp\left(\frac{q + \lambda A^*(A^*(A))^{-\zeta}}{\frac{A^*(A)}{A} - 1} \cdot \frac{\theta}{r}\right)$$

It is clear that the LHS is strictly increasing and the RHS is strictly decreasing in $A^*(A)/A$. All the comparative static results follow from perturbing parameters by some $\varepsilon > 0$ and finding the compensating variation in $A^*(A)$ that leaves the (FE) satisfied.

7.0.6 Aggregate Outcomes in Kortum (1997)

Distributions: Under full enforcement and zero fixed cost of entry, the distributions have closed form expressions as derived in Kortum (EMA 1997). Below I present the distributions over inventive steps, Γ_{ξ}, over supplanted technologies, $\Gamma_{A_{-1}}$, and their joint distribution.

$$\Gamma_{\xi}(\xi) = 1 - \xi^{-\zeta}$$

$$\Gamma_{A_{-1}}(A_{-1}) = \frac{\left(1 + \lambda A_{-1}^{-\zeta}\right) e^{-\lambda A_{-1}^{-\zeta}} - e^{-\lambda}}{1 - e^{-\lambda}}$$

$$\Gamma(\xi, A_{-1}) = \Gamma_{\xi}(\xi)\Gamma_{A_{-1}}(A_{-1})$$

Strikingly, ξ and A_{-1} are independent under the assumptions underlying the Pareto and Frechet distributions. This depends crucially on the fact that Γ_{ξ} is independent of A_{-1} since the cutoff productivity is uniformly $A^* = 1$ across A_{-1}'s. This follows from $I = 0$.

Rental Price: Substituting the final good producer's optimality condition into the definition of final good supply yields the rental price of capital:

$$Y = \exp\left(\int_0^1 \ln\left(A_{-1}(\omega)\frac{Y}{r}\right) d\omega\right) = \exp\left(\int_1^\infty \ln(A_{-1}) d\Gamma_{A_{-1}}(A_{-1})\right) \cdot \frac{Y}{r}$$

Therefore the rental price is given by $r = \overline{A_{-1}}$, where $\overline{A_{-1}}$ is the aggregate productivity of supplanted technologies.

Final Output: The capital market clearing condition and final demand function yield the following expression for aggregate output:

$$k = \int_0^1 K'(\omega) d\omega \qquad \Longrightarrow \qquad Y = \frac{rk}{\int_1^\infty \xi^{-1} d\Gamma_{\xi}(\xi)} \equiv \frac{\overline{A}}{\Xi}\, k$$

where $\overline{A}/\Xi$ is a measure of the aggregate productivity of frontier technologies. The relationship is clearly linear, as in the neoclassical growth model with a single factor of production.

7.0.7 Aggregate Outcomes under Limited Enforcement

Distributions: Under limited enforcement, the distributions are similar to those in Kortum (1997) except now must take into account the cutoff productivity rule. First, the distribution of ideas is $F(A) = 1 - A^{-\zeta}$ and the productivity distribution over incumbent firms is $G(A) = exp(-\lambda A^*(A)^{-\zeta})$. Below I present the distribution over supplanted technologies, $\Gamma_{A_{-1}}(\cdot)$, the conditional distribution over inventive steps, $\Gamma_{\xi}(\cdot|A_{-1})$, and their joint distribution.

$$\Gamma_{A_{-1}}(A_{-1}) = \frac{\left(1 + \lambda A^*(A_{-1})^{-\zeta}\right) e^{-\lambda A^*(A_{-1})^{-\zeta}} - e^{-\lambda}}{1 - e^{-\lambda}}$$

$$\Gamma_{\xi}(\xi|A_{-1}) = 1 - \left(\frac{\xi}{\xi^*(A_{-1})}\right)^{-\zeta}$$

$$\Gamma(\xi, A_{-1}) = \Gamma_{A_{-1}}(A_{-1}) - \left(\frac{\xi}{\Lambda(A_{-1})}\right)^{-\zeta}$$

where

$$\xi^*(A) \equiv \frac{A^*(A)}{A}$$

$$\Lambda(A_{-1})^{\zeta} = \frac{\lambda}{1 - e^{-\lambda}} \int^{A_{-1}} a^{-\zeta} dG(a)$$

Strikingly, much like Kortum's environment, $\Gamma_{\xi}(\xi|A_{-1})$ is a Pareto distribution - except now conditional on the cutoff rule.

Rental Price: The rental price takes the same form as in Kortum (1997), except here the distribution over A_{-1} is different.

Final Output: Let $K^F(\omega)$ denote the "follower" firm's capital stock in sector ω. The capital market clearing condition and final demand function yield the following expression for aggregate output:

$$k = \int_0^1 \Big(K(\omega) + K^F(\omega)\Big) d\omega$$

$$rk = \int_0^1 rK_t(\omega)d\omega + \left(Y - \int_0^1 \xi(\omega) rK_t(\omega)d\omega\right)$$

$$rk = Y - \int_0^1 (\xi(\omega) - 1) rK_t(\omega)d\omega$$

$$Y = rk + \int_0^1 S(\omega)d\omega$$

where the last term is the aggregate project surplus.

To characterize aggregate surplus:

$$\int_0^1 S(\omega)d\omega = \int_1^{\infty} \int_{\xi^*(A_{-1})}^{\infty} \left\{ \int_0^{T(A,\xi)} (\xi - 1)\frac{V_0(A,\xi)}{\theta} e^{qt} dt + \int_{T(A,\xi)} \infty (1 - \xi^{-1}) Y e^{-\beta(A)t} dt \right\} \gamma(\xi, A_{-1}) d\xi dA_{-1}$$

$$= Y \int_1^{\infty} \int_{\xi^*(A_{-1})}^{\infty} \frac{\delta(A)i}{\phi(A,\xi)} \left\{ \left(\frac{1}{q} + \frac{1}{\beta(A)}\right) \left(\frac{\phi(A,\xi)}{i} \cdot \frac{1 - \xi^{-}1}{\delta(A)}\right)^{q/\delta(A)} - \frac{1}{q} \right\} \gamma(\xi, A_{-1}) d\xi dA_{-1}$$

$$\equiv s(\theta) Y$$

where $s(\theta)$ can be interpreted as the share of output accruing to financing firms. As such we would expect this share to be larger in economies with lower financial frictions (e.g. lower θ).

Therefore, substituting aggregate surplus back into the expression for aggregate output, we obtain:

$$Y = \frac{r}{1 - s(\theta)} k$$