Advanced Topics in Macroeconomics: Quantitative Macroeconomics

Problem Set 1
Due February 27, 2018

Question 1: (Wage Growth and Liquidity Constraints) Suppose an agent lives for three periods. The agent has additively separable preferences over consumption and hours worked given by:

$$U(c,h) = u(c) - v(h) = \frac{c^{1-\sigma}}{1-\sigma} - \frac{h^{1+\frac{1}{\gamma}}}{1+\frac{1}{\gamma}} \quad \text{such that} \quad \sigma > 1, \gamma > 0$$

and discounts future utility with discount factor β. The agent is endowed with a unit of time $h_t \in [0,1]$ for $t = 1,2$. The agent's labor productivity in period 1 is e_1, in period 2 is $e_2 \equiv ge_1$ and in period 3 is $e_3 = 0$. Let labor productivity growth be positive, $g > 1$. The agent receives zero initial endowment of wealth, $a_1 = 0$, but can save in periods 1 and 2 with given interest $R \equiv (1+r)$. Agents cannot borrow ($a_2, a_3 \geq 0$). Accordingly, we can write the agent's utility maximization problem as:

$$\max_{(c_1,c_2,c_3,a_2,a_3,h_1,h_2)} [u(c_1) - v(h_1)] + \beta[u(c_2) - v(h_2)] + \beta^2 u(c_3)$$

$$\begin{aligned}
\text{s.t.} \quad c_1 + a_2 &\leq we_1h_1 \\
c_2 + a_3 &\leq we_2h_2 + Ra_2 \\
c_3 &\leq Ra_3 \\
h_1, h_2 &\in [0,1] \\
c_t, a_{t+1} &\geq 0 \quad \forall\, t
\end{aligned}$$

(a) Assume that the agent chooses $a_2 = 0$. Characterize the resulting solution for consumption and hours, $(c_1, c_2, c_3, h_1, h_2)$, and second period savings (a_3) as a function of prices and parameters. Assume that hours decisions h_1, h_2 are interior to the set $[0,1]$. (Hint: take first order conditions, substitute $a_2 = 0$, and rewrite. It may simplify notation to define $\kappa \equiv \beta^{-\frac{1}{\sigma}} R^{1-\frac{1}{\sigma}}$.)

(b) Prove that if g is sufficiently large then $a_2 = 0$ is the agent's optimal choice of first period saving. (Hint: prove by contradiction – suppose that $a_2 > 0$ for any value of g and show that there must be a value for g for which $a_2 = 0$ or else the savings decision would not be optimal.)

Question 2: (Gale 1967) Suppose there is a finitely lived agent such that $t = 0, 1, \ldots, T$. There is a single consumption good and the agent's instantaneous utility function over consumption is $u(c) = \log(c)$. The agent's discount factor is $\beta < 1$. Suppose the agent is endowed with s_0 units of the consumption good and has access to a storage technology tht takes s units of period t consumption goods and transforms it into s units of period $t+1$ consumption goods. Each period the agent chooses how much of his endowment to consume or to save (using the storage technology) for the next period.

(a) Write the agent's dynamic optimization problem in either sequential or recursive form.

(b) Solve for the agent's optimal paths of consumption and savings $\{c_t, s_{t+1}\}_{t=0}^{T}$ given the endowment s_0. Your answers should be analytical expressions. (Hint: $s_{T+1} = 0$.)

Question 3: (Cass-Koopmans Growth Model) Consider an infinite horizon economy, $t = 0, 1, 2, \ldots$. There is a single consumption good. There exists a representative agent who values the consumption good according to an instantaneous utility function $u(c)$ and discounts time with factor $\beta < 1$. The representative agent has access to an investment technology that transforms k units of the consumption good today into k^α units of the consumption good tomorrow. Assume that $\alpha < 1$. Each period, the representative agent chooses how much to consume (c) and how much to invest (k'). In sequential form, the Social Planner's problem is:

$$\max_{\{c_t, k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t)$$

$$\begin{aligned} \text{s.t.} \quad c_t + k_{t+1} &\leq k_t^\alpha + (1-\delta)k_t \\ c_t, k_{t+1} &\geq 0 \\ \text{given} \quad & k_0 \end{aligned}$$

(a) Write the Bellman equation. (If you need a refresher, see the Ljungqvist and Sargent textbook listed in the syllabus.)

(b) Assume that $u(c) = \log(c)$ and $\delta = 1$ and solve the dynamic program in (a). To obtain a solution, guess and verify that $v(k) = A + B\log(k)$ and solve for A and B. Your answer should be an analytical expression for the value function $v(k)$, the consumption policy function $c(k)$ and the capital savings policy function $k'(k)$.