Notes on Mechanism Design and Public Economics

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Contents

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1 Bayesian Games and Mechanism Design

In this chapter I will state and prove a few of the fundamental results of mechanism design. I will use Malaith and Postlewaite's (1990) environment as a basis on which to discuss the Revelation Principle, Myerson's characterization of incentive compatibility in linear environments and the Myerson-Satterthwaite characterization theorem. I will then discuss Mailath and Postlewaite's asymptotic inefficiency result. I will then apply these results to the environment developed in Rob (1989) and demonstrate how the outcomes in the public goods environment are technically similar to the outcomes in a negative externality environment. Lastly I will discuss the examples in Chari and Jones (2001) which make clear the similarities between these two environments.

1.1 The Revelation Principle: Myerson (EMA 1979)

Consider an environment with N agents who receive a random type θ_i drawn from the set Θ_i with probability density given by $f(\theta)$, where $\theta = (\theta_1, \theta_2, \ldots, \theta_N)$. We say $\theta \in \Theta \equiv \Theta_1 \times \Theta_2 \times \cdots \times \Theta_N$. Denote $\theta_{-i} = (\theta_1, \ldots, \theta_{i-1}, \theta_{i+1}, \ldots, \theta_N)$.

Agents must make a collective decision from a set X of possible alternative decisions. After nature chooses θ, each agent chooses an action a_i from the action set A_i. Denote $A \equiv A_1 \times \cdots \times A_N$. An outcome of individual decision making is a function $x : A \to X$. The payoff to an agent i is a von Neumann-Morgenstern utility function $u_i(x, \theta_i)$.

In this setting, the agents' actions can be interpreted as reports of their type θ. In what follows I will call Θ the *type space*, and A the *action or message space*.

Definition (Social Choice Function) A *social choice function* is a function $y : \Theta \to X$ that assigns a collective choice for any report of θ.

Definition (Mechanism) A *mechanism* consists of strategy sets for each of N agents, $(A_i)_{i=1}^N$, and outcome function $x : A \to X$. Denote the mechanism as $\Gamma = \{(A_i)_{i=1}^N, x(a)\}$.

Definition (Bayesian Nash Equilibrium) The strategy profile $\{a_i(\theta_i)\}_{i=1}^N$ is a *Bayesian Nash equilibrium of the mechanism* $\Gamma = \{(A_i)_{i=1}^N, x\}$ if for all $i, \theta \in \Theta$, and $\widehat{a}_i$

$$\mathbb{E}_{-i}\Big[u_i\Big(x(a_i(\theta_i), a_{-i}(\theta_{-i})), \theta_i\Big)\Big|\theta_i\Big] \geq \mathbb{E}_{-i}\Big[u_i\Big(x(\widehat{a}_i, a_{-i}(\theta_{-i})), \theta_i\Big)\Big|\theta_i\Big] \tag{1}$$

where, for some arbitrary function $g : \Theta \to X$, the above expectations operator takes the form

$$\mathbb{E}_{-i}\Big[g(\theta)\Big] = \int_{\theta_{-i}} g(\theta_{-i}|\theta_i) f(\theta_{-i}|\theta_i) d\theta_i$$

and where $f(\theta_{-i}|\theta_i)$ is the conditional probability of nature choosing the vector of types θ_{-i} for all agents other than i given that i drew θ_i.

Definition (Implementation) A social choice function $y : \Theta \to X$ is *implementable* in Bayesian Nash Equilibrium if there exists a Bayesian Nash equilibrium, $\{a_i(\theta_i)\}_{i=1}^N$, of the mechanism $\Gamma = \{(A_i)_{i=1}^N, x(a)\}$ such that $x(a(\theta)) = y(\theta)$ for all $\theta \in \Theta$.

Definition (Direct Revelation Mechanism) A *direct revelation mechanism* consists of strategy sets for each of N agents, $(A_i)_{i=1}^N$, and outcome function $x : \Theta \to X$ such that $A_i = \Theta_i$ for all i. Denote the direct revelation mechanism as $\Gamma_R = \{(\Theta_i)_{i=1}^N, x(\theta)\}$.

Definition (Truthful Implementation) An outcome function $y : \Theta \to X$ is *truthfully implementable or Bayesian incentive compatible* if for all $\theta \in \Theta$ and $i = 1, \ldots, N$, $a_i(\theta_i) = \theta_i$ is a Bayesian Nash equilibrium of the direct revelation mechanism $\Gamma_R = \{(\Theta_i)_{i=1}^N, y(\theta)\}$..

Theorem 1.1.1 (Revelation Principle)
If a mechanism $\{(A_i)_{i=1}^N, x(a)\}$ implements the social choice function $y : \Theta \to X$, then y is truthfully implementable as a Bayesian Nash Equilibrium of the direct revelation mechanism $\Gamma_R = \{(\Theta_i)_{i=1}^N, x(\theta)\}$.

Proof. If the mechanism implements y, then there exist $\{a_i(\theta_i)\}_{i=1}^N$ such that $x(a(\theta)) = y(\theta)$ for all θ, and for all $i, \theta \in \Theta$, and $\widehat{a}_i$ equation (1) holds.

$$\mathbb{E}_{-i}\Big[u_i\Big(x(a_i(\theta_i), a_{-i}(\theta_{-i})), \theta_i\Big)\Big|\theta_i\Big] \geq \mathbb{E}_{-i}\Big[u_i\Big(x(\widehat{a}_i, a_{-i}(\theta_{-i})), \theta_i\Big)\Big|\theta_i\Big]$$

If for all $i, \theta \in \Theta$, and $\widehat{a}_i$ we have (1) holding, then for all $i, \theta \in \Theta$, and $\widehat{\theta}_i \in \Theta_i$

$$\mathbb{E}_{-i}\Big[u_i\Big(x(a_i(\theta_i), a_{-i}(\theta_{-i})), \theta_i\Big)\Big|\theta_i\Big] \geq \mathbb{E}_{-i}\Big[u_i\Big(x(a_i(\widehat{\theta}_i), a_{-i}(\theta_{-i})), \theta_i\Big)\Big|\theta_i\Big]$$

But since $x(a(\theta)) = y(\theta)$ for all θ, then for all $i, \theta \in \Theta$, and $\widehat{\theta}_i \in \Theta_i$

$$\mathbb{E}_{-i}\Big[u_i\Big(y(\theta_i, \theta_{-i}), \theta_i\Big)\Big|\theta_i\Big] \geq \mathbb{E}_{-i}\Big[u_i\Big(y(\widehat{\theta}_i, \theta_{-i}), \theta_i\Big)\Big|\theta_i\Big]$$

Therefore, y is truthfully implementable in Bayesian Nash equilibrium. □

In the next section I will introduce a public goods environment and present several fundamental results in mechanism design.

1.2 Public Good Provision: Mailath and Postlewaite (RES 1990)

Mailath and Postlewaite (1990) analyze the classic public goods problem under private information. Similar to the environment described in section 1.1, agents stochastically receive a privately known valuation, $\theta_i \in \Theta_i \equiv [\underline{\theta}_i, \overline{\theta}_i]$, of a public good that costs $C(N)$. Here, cost depends on the number of agents, N, that potentially contribute to the good. Valuations (types) are independently drawn from the CDF $F_i(\theta)$ for each agent, with full support over Θ_i. Assume now that the type space is a continuum for each agent, $\Theta_i = [\underline{\theta}_i, \overline{\theta}_i]$, and that types are independent across agents (e.g. $f(\theta) = \prod_{i=1}^N f_i(\theta_i)$ with f_i having full support over Θ_i for each i).

Definition (Mechanism) A **mechanism** in this environment consists of agents' strategies $(A_i)_{i=1}^N$, the probability of public provision $\rho : \Theta \to [0, 1]$, and a transfer scheme $(\xi_i)_{i=1}^N$ such that $\xi : \Theta \to \mathbb{R}^N$.

Suppose that each agent has a von Neumann-Morgenstern utility function that is linear in type:

$$u_i(\theta_i, \theta_{-i}) = \Big(\theta_i - \xi_i(\theta_i, \theta_{-i})\Big)\rho(\theta_i, \theta_{-i}) = \theta_i\rho(\theta) - \xi_i(\theta)\rho(\theta)$$

For notational convenience, define the following:

$$\overline{\rho}_i(\widehat{\theta}_i) = \mathbb{E}_{-i}[\rho(\widehat{\theta}_i, \theta_{-i})]$$

$$\overline{\xi}_i(\widehat{\theta}_i) = \mathbb{E}_{-i}[\rho(\widehat{\theta}_i, \theta_{-i})\xi(\widehat{\theta}_i, \theta_{-i})]$$

where $\overline{\rho}_i$ is the agent's expectation of the probability of provision given his action (announcing type $\widehat{\theta}_i$) and $\overline{\xi}_i$ is his expected transfer given his action. Accordingly, an agent's expected utility from reporting type $\widehat{\theta}_i$ conditional on being type θ_i is given by:

$$w_i(\widehat{\theta}_i, \theta_i) \equiv \mathbb{E}_{-i}\Big[\Big(\theta_i - \xi_i(\widehat{\theta}_i, \theta_{-i})\Big)\rho(\widehat{\theta}_i, \theta_{-i})\Big] = \theta_i\overline{\rho}_i(\widehat{\theta}_i) - \overline{\xi}_i(\widehat{\theta}_i)$$

Lastly, denote $U_i(\theta_i) \equiv w_i(\theta_i, \theta_i)$ as the expected utility from truthful reporting. Assume that U_i is differentiable. Notice that expected utility is also linear in type.

1.2.1 Myerson (MOR 1981): IC with Linear Utility

The following theorem provides necessary and sufficient conditions for a social choice function to be Bayesian incentive compatible in this general class of environments.

Theorem 1.2.1 (Characterization Theorem for IC in Linear Economies) A mechanism $\{(A_i)_{i=1}^N, k(\theta), (\xi_i(\theta))_{i=1}^N\}$ is Bayesian incentive compatible if and only if

(i) $\overline{\rho}_i(\theta_i)$ is nondecreasing in θ_i

(ii) $U_i(\theta_i) = U_i(\widehat{\theta}_i) + \displaystyle\int_{\widehat{\theta}_i}^{\theta_i} \overline{\rho}_i(x)dx \quad \forall\, \widehat{\theta}_i, \theta_i$

Proof. (a) *Necessity:* First notice that we can write the expected payoff as:

$$\begin{aligned} w_i(\widehat{\theta}_i, \theta_i) &= \theta_i\overline{\rho}_i(\widehat{\theta}_i) - \overline{\xi}_i(\widehat{\theta}_i) \\ &= \theta_i\overline{\rho}_i(\widehat{\theta}_i) - \overline{\xi}_i(\widehat{\theta}_i) + \widehat{\theta}_i\overline{\rho}_i(\widehat{\theta}_i) - \widehat{\theta}_i\overline{\rho}_i(\widehat{\theta}_i) \\ &= \widehat{\theta}_i\overline{\rho}_i(\widehat{\theta}_i) - \overline{\xi}_i(\widehat{\theta}_i) + \theta_i\overline{\rho}_i(\widehat{\theta}_i) - \widehat{\theta}_i\overline{\rho}_i(\widehat{\theta}_i) \\ &= U_i(\widehat{\theta}_i) + (\theta_i - \widehat{\theta}_i)\overline{\rho}_i(\widehat{\theta}_i) \end{aligned}$$

If the mechanism is incentive compatible then for all types $\widehat{\theta}_i > \theta_i$ the following two inequalities are satisfied:

$$\begin{aligned} U_i(\theta_i) &\geq w_i(\widehat{\theta}_i, \theta_i) \equiv U_i(\widehat{\theta}_i) + (\theta_i - \widehat{\theta}_i)\overline{\rho}_i(\widehat{\theta}_i) \\ U_i(\widehat{\theta}_i) &\geq w_i(\theta_i, \widehat{\theta}_i) \equiv U_i(\theta_i) + (\widehat{\theta}_i - \theta_i)\overline{\rho}_i(\theta_i) \end{aligned}$$

These inequalities can be rewritten, respectively, as:

$$\left.\begin{array}{lcl}\overline{\rho}_i(\widehat{\theta}_i) & \geq & \dfrac{U_i(\widehat{\theta}_i) - U_i(\theta_i)}{\widehat{\theta}_i - \theta_i} \\ \overline{\rho}_i(\theta_i) & \leq & \dfrac{U_i(\widehat{\theta}_i) - U_i(\theta_i)}{\widehat{\theta}_i - \theta_i}\end{array}\right\} \implies \overline{\rho}_i(\widehat{\theta}_i) \geq \frac{U_i(\widehat{\theta}_i) - U_i(\theta_i)}{\widehat{\theta}_i - \theta_i} \geq \overline{\rho}_i(\theta_i)$$

Therefore, $\overline{\rho}_i(\theta_i)$ is nondecreasing since we took $\widehat{\theta}_i > \theta_i$. Furthermore, since we assumed that U_i is differentiable, if we take the limit of the above expression as $\widehat{\theta}_i \to \theta_i$ we obtain

$$U_i'(\theta_i) = \lim_{\widehat{\theta}_i \to \theta_i} \frac{U_i(\widehat{\theta}_i) - U_i(\theta_i)}{\widehat{\theta}_i - \theta_i} = \overline{\rho}_i(\theta_i)$$

Taking the integral of the above equation over the interval $[\widehat{\theta}_i, \theta_i]$ and applying the fundamental theorem of calculus, we obtain:

$$\int_{\widehat{\theta}_i}^{\theta_i} U_i'(x)\ dx \ = \ U_i(\theta_i) - U_i(\widehat{\theta}_i) \ = \ \int_{\widehat{\theta}_i}^{\theta_i} \overline{\rho}_i(x)\ dx$$

(b) *Sufficiency:* Suppose (i) and (ii) are satisfied. Consider some $\widehat{\theta}_i, \theta_i$ such that $\theta_i > \widehat{\theta}_i$. Then,

$$\begin{aligned} U_i(\theta_i) - U_i(\widehat{\theta}_i) &= \int_{\widehat{\theta}_i}^{\theta_i} \overline{\rho}_i(x)\ dx && \text{[by (ii)]} \\ &\geq \int_{\widehat{\theta}_i}^{\theta_i} \overline{\rho}_i(\widehat{\theta}_i)\ dx && \text{[by (i) and } \theta_i > \widehat{\theta}_i\text{]} \\ &= (\theta_i - \widehat{\theta}_i)\overline{\rho}_i(\widehat{\theta}_i) \end{aligned}$$

Then, since we chose $\widehat{\theta}_i, \theta_i$ arbitrarily, we can derive:

$$\begin{aligned} U_i(\theta_i) &\geq U_i(\widehat{\theta})_i + (\theta_i - \widehat{\theta}_i)\overline{\rho}_i(\widehat{\theta}_i) \equiv w_i(\widehat{\theta}_i, \theta_i) \\ U_i(\widehat{\theta}_i) &\geq U_i(\theta_i) + (\widehat{\theta}_i - \theta_i)\overline{\rho}_i(\theta_i) \equiv w_i(\theta_i, \widehat{\theta}_i) \end{aligned}$$

Therefore the revelation mechanism is incentive compatible. □

Hence, this theorem gives us a method for identifying incentive compatible mechanisms for linear environments.

1.2.2 Myerson and Satterthwaite (JET 1983): Participation

In the previous sections, 1.1 and 1.2.1, we made an important implicit assumption that agents must participate in the mechanism. However, let's consider a more general environment in which agents have an outside option and are allowed to choose not to participate in the mechanism.

We will define three classes of participation constraints, each corresponding to *when* agents decide to accept or reject the mechanism relative to agents' resolution of uncertainty about their type. That is, the mechanism guarantees each agent a contingent allocation, which an agent may reject depending on his information set. Let's define three different participation constraints in order to fix ideas.

First consider the *ex post participation constraint*, which requires that a mechanism allocate enough resources to the agent to ensure their participation $after$ the agents have revealed their type (e.g. type is then public information).

$$u_i\Big(y(\theta_i, \theta_{-i}), \theta_i\Big) \geq V_i^{aut}(\theta_i) \qquad \forall\ \theta_i, \theta_{-i}$$

where $y : \Theta \to X$ is an outcome function, u_i is again agent i's utility function, and V^{aut} is the value to the agent of rejecting the mechanism and engaging in private consumption. I call this the value of autarky.

Next, consider the *interim participation constraint*, which requires the mechanism to ensure participation after all agents know their own type but before they reveal their type to others.

$$\mathbb{E}_{-i}\Big[u_i\Big(y(\theta_i, \theta_{-i}), \theta_i\Big)\Big|\theta_i\Big] \geq V_i^{aut}(\theta_i) \qquad \forall\ \theta_i$$

Lastly, consider the *ex ante participation constraint*, which requires the mechanism to ensure participation prior to the realization of any uncertainty by any agent.

$$\mathbb{E}_{\theta}\Big[u_i\Big(y(\theta_i, \theta_{-i}), \theta_i\Big)\Big] \geq \mathbb{E}_{\theta_i}\Big[V_i^{aut}(\theta_i)\Big]$$

Notice that any mechanism that satisfies ex post participation will satisfy interim and ex ante participation. Furthermore any mechanism that satisfies interim participation will satisfy ex ante participation. Accordingly ex post participation places the most stringent constraints on the principal.

Myerson and Satterthwaite (1983) have provided a set of conditions where, in fact, no mechanism can truthfully implement any "efficient" social choice function $y : \Theta \to X$ in the presence of interim participation constraints. Following Myerson and Satterthwaite (1983) I will first provide necessary and sufficient conditions for a social choice function to be Bayesian incentive compatible in the presence of interim participation constraints. I will then show that a property of this characterization is that no ex post efficient social choice function is truthfully implementable. First I define ex post efficiency.

Definition (Ex Post Efficiency) A social choice function $y : \Theta \to X$ is *ex post efficient* if for any $\theta \in \Theta$ there does not exist a $x \in X$ such that

(a) $u_i(x, \theta_i) \geq u_i(y(\theta), \theta_i)$ for every agent, i, **and**

(b) $u_i(x, \theta_i) > u_i(y(\theta), \theta_i)$ for some agent i

This definition can be interpreted to mean that the social choice function allocates resources to agents, for any realization of types, in a Pareto optimal manner.

For the following theorems, consider the environment from section 1.2.1. Furthermore, denote the CDF over the type space as $F_i(\theta_i)$ for each agent i. And let the value of autarky be zero (e.g. $V_i^{aut}(\theta_i) = 0$). Lastly, we will require that the transfer rule satisfy the following property:

Definition (Ex Ante Balanced Budget) A transfer rule $(\xi_i(\theta))_{i=1}^N$ satisfies *ex ante budget balancedness* if:

$$\mathbb{E}_\theta\left[\left(\sum_{i=1}^N \xi_i(\theta) - C(N)\right)\rho(\theta)\right] \geq 0$$

That is, before agents acquire any information about their type, the mechanism ensures transfers will cover the cost of provision and will not require an infusion of outside resources.

Lastly, consider the following cost function: $C(N) = cN$. In this function, $c > 0$ is the per person cost. I consider this cost function in order to simplify the proofs of the following results.

Theorem 1.2.2 (Characterization Theorem Under Voluntary Participation) Suppose $\overline{\rho}_i(\cdot)$ is increasing for each agent i. There exists a transfer scheme $(\xi_i(\theta))_{i=1}^N$ such that a mechanism $\{(\Theta_i)_{i=1}^N, \rho(\theta), (\xi_i(\theta))_{i=1}^N\}$ satisfies incentive compatibility, interim voluntary participation and ex ante budget balancedness if and only if:

$$\mathbb{E}_\theta\left[\left(\sum_{i=1}^N \left\{\theta_i - \frac{1-F_i(\theta_i)}{f_i(\theta_i)}\right\} - cN\right)\rho(\theta)\right] \geq 0 \qquad (\star)$$

Proof. (a) *Necessity:* To start the proof, I will show that any incentive compatible mechanism will satisfy equality between $\sum_i U_i(\underline{\theta}_i) - \mathbb{E}_\theta[\sum_i \xi_i^*(\theta)]$ and the LHS of ($\star$). We will then show that the conditions of the proof will guarantee the LHS of ($\star$) is nonnegative.

For any agent i, ex ante expected transfers to other agents can be written as:

$$\begin{aligned}
\mathbb{E}_\theta\Big[\,(\xi_i(\theta) - c)\,\rho(\theta)\Big] &= \mathbb{E}_i\Big[\overline{\xi}_i(\theta_i) - c\overline{\rho}_i(\theta_i)\Big] \\
&= \mathbb{E}_i\Big[\theta_i\overline{\rho}_i(\theta_i) - U_i(\theta_i) - c\overline{\rho}_i(\theta_i)\Big] && \text{By DEF of } U_i \\
&= \mathbb{E}_i\left[(\theta_i - c)\overline{\rho}_i(\theta_i) - U_i(\underline{\theta}_i) - \int_{\underline{\theta}_i}^{\theta_i} \overline{\rho}_i(\tilde{\theta})\ d\tilde{\theta}\right] && \text{By THM 1.2.1} \\
&= \mathbb{E}_i\left[(\theta_i - c)\overline{\rho}_i(\theta_i) - \int_{\underline{\theta}_i}^{\theta_i} \overline{\rho}_i(\tilde{\theta})\ d\tilde{\theta}\right] - U_i(\underline{\theta}_i)
\end{aligned}$$

Next, integrate by parts the second term in the RHS expectation (in the last equality above). To do so, we will use the following definitions:

$$g_1(\theta_i) = \int_{\underline{\theta}_i}^{\theta_i} \overline{\rho}_i(\tilde{\theta})d\tilde{\theta} \qquad \text{and} \qquad g_2'(\theta_i) = f_i(\theta_i)$$

where, $\int g_1(x)g_2'(x)dx = g_1(x)g_2(x) - \int g_1'(x)g_2(x)dx$. To find the derivative of $g_1(\cdot)$ we must use Leibniz's Rule:

$$\frac{\partial}{\partial\theta_i}\int_{\underline{\theta}_i}^{\theta_i} \overline{\rho}_i(\theta)d\theta \quad = \quad \int_{\underline{\theta}_i}^{\theta_i} \frac{\partial}{\partial\theta_i}\overline{\rho}_i(\theta)d\theta + \overline{\rho}_i(\theta_i)\frac{\partial\theta_i}{\partial\theta_i} - \overline{\rho}_i(\underline{\theta}_i)\frac{\partial\underline{\theta}_i}{\partial\theta_i}$$

$$= 0 \cdot \int_{\underline{\theta}_i}^{\theta_i} d\theta + \overline{\rho}_i(\theta_i) \cdot 1 - \overline{\rho}_i(\underline{\theta}_i) \cdot 0$$

$$= \overline{\rho}_i(\theta_i)$$

Accordingly, the integration by parts will yield:

$$\mathbb{E}_i \left[\int_{\underline{\theta}_i}^{\theta_i} \overline{\rho}_i(\tilde{\theta}) d\tilde{\theta} \right] = \int_{\underline{\theta}_i}^{\overline{\theta}_i} \left(\int_{\underline{\theta}_i}^{\theta_i} \overline{\rho}_i(\tilde{\theta}) d\tilde{\theta} \right) f_i(\theta_i) d\theta_i$$

$$= \left[\int_{\underline{\theta}_i}^{\theta_i} \overline{\rho}_i(\tilde{\theta}) d\tilde{\theta} \cdot F_i(\theta_i) \right]_{\underline{\theta}_i}^{\overline{\theta}_i} - \int_{\underline{\theta}_i}^{\overline{\theta}_i} \overline{\rho}_i(\theta_i) F_i(\theta_i) d\theta_i$$

$$= \left[\int_{\underline{\theta}_i}^{\overline{\theta}_i} \overline{\rho}_i(\theta_i) d\theta_i \cdot 1 - 0 \right] - \int_{\underline{\theta}_i}^{\overline{\theta}_i} \overline{\rho}_i(\theta_i) F_i(\theta_i) d\theta_i$$

$$= \int_{\underline{\theta}_i}^{\overline{\theta}_i} \overline{\rho}_i(\theta_i)(1 - F_i(\theta_i)) d\theta_i$$

Substituting the integral back into expected transfers, we obtain:

$$\mathbb{E}_\theta \Big[(\xi_i(\theta) - c)\, \rho(\theta) \Big] = \mathbb{E}_i \left[(\theta_i - c)\overline{\rho}_i(\theta_i) - \int_{\underline{\theta_i}}^{\theta_i} \overline{\rho}_i(\tilde{\theta})\ d\tilde{\theta} \right] - U_i(\underline{\theta}_i)$$

$$= \mathbb{E}_i \Big[(\theta_i - c)\overline{\rho}_i(\theta_i) \Big] - \int_{\underline{\theta}_i}^{\overline{\theta}_i} \overline{\rho}_i(\theta_i)(1 - F_i(\theta_i)) d\theta_i - U_i(\underline{\theta}_i)$$

$$= \mathbb{E}_i \left[(\theta_i - c)\overline{\rho}_i(\theta_i) - \overline{\rho}_i(\theta_i) \frac{1 - F_i(\theta_i)}{f_i(\theta_i)} \right] - U_i(\underline{\theta}_i)$$

$$= \mathbb{E}_i \left[\left(\theta_i - \frac{1 - F_i(\theta_i)}{f_i(\theta_i)} - c \right) \overline{\rho}_i(\theta_i) \right] - U_i(\underline{\theta}_i)$$

Which we can rewrite as the expectation over θ:

$$\mathbb{E}_\theta \Big[(\xi_i(\theta) - c)\, \rho(\theta) \Big] = \mathbb{E}_\theta \left[\left(\theta_i - \frac{1 - F_i(\theta_i)}{f_i(\theta_i)} - c \right) \rho(\theta) \right] - U_i(\underline{\theta}_i)$$

$$= \int_{\underline{\theta}_1}^{\overline{\theta}_1} \cdots \int_{\underline{\theta}_N}^{\overline{\theta}_N} \left(\theta_i - \frac{1 - F_i(\theta_i)}{f_i(\theta_i)} - c \right) \rho(\theta) \prod_{j=1}^{N} f_j(\theta_j) d\theta_j - U_i(\underline{\theta}_i)$$

Finally by summing over all agents and rearranging we can write ($\star$):

$$\mathbb{E}_\theta \left[\left(\sum_{i=1}^{N} \left\{ \theta_i - \frac{1 - F_i(\theta_i)}{f_i(\theta_i)} \right\} - cN \right) \rho(\theta) \right] = \underbrace{\sum_{i=1}^{N} U_i(\underline{\theta}_i)}_{\geq 0 \text{ by VP}} + \underbrace{\mathbb{E}_\theta \left[\left(\sum_{i=1}^{N} \xi_i(\theta) - cN \right) \rho(\theta) \right]}_{\geq 0 \text{ by BB}} \geq 0$$

Thus, ($\star$) is nonnegative since the voluntary participation keeps the first RHS term nonnegative and budget balancedness keeps the second RHS term nonnegative.

(b) *Sufficiency:* Now assume that ($\star$) is satisfied. To show that there exists a transfer scheme that satisfies incentive compatibility, interim voluntary participation and ex ante budget balancedness, consider the following construction:

$$\overline{\xi}_i(\theta_i) = a_i + \theta_i\overline{\rho}_i(\theta_i) - \int_{\underline{\theta}_i}^{\theta_i} \overline{\rho}_i(\theta)d\theta - \frac{1}{n-1}\mathbb{E}_{-i}\left[\sum_{j\neq i}\left((\theta_j - c)\overline{\rho}_j(\theta_j) - \int_{\underline{\theta}_j}^{\theta_j}\overline{\rho}_j(\theta)d\theta\right)\right]$$

We will find conditions on the constant a_i that satisfy incentive compatibility, interim voluntary participation and ex ante budget balancedness.

In order for the transfer scheme to satisfy ex ante budget balancedness

$$\begin{aligned}
\mathbb{E}_\theta\left[\left(\sum_{i=1}^N \xi_i(\theta) - cN\right)\rho(\theta)\right] &= \sum_{i=1}^N a_i + \mathbb{E}_\theta\left[\sum_{i=1}^N\left((\theta_i - c)\overline{\rho}_i(\theta_i) - \int_{\underline{\theta}_i}^{\theta_i}\overline{\rho}_i(\theta)d\theta\right)\right] \\
&\quad - \frac{1}{n-1}\mathbb{E}_\theta\left[\sum_{i=1}^N\sum_{j\neq i}\left((\theta_j - c)\overline{\rho}_j(\theta_j) - \int_{\underline{\theta}_j}^{\theta_j}\overline{\rho}_j(\theta)d\theta\right)\right] \\
&= \sum_{i=1}^N a_i + \mathbb{E}_\theta\left[\sum_{i=1}^N\left((\theta_i - c)\overline{\rho}_i(\theta_i) - \int_{\underline{\theta}_i}^{\theta_i}\overline{\rho}_i(\theta)d\theta\right)\right] \\
&\quad - \mathbb{E}_\theta\left[\sum_{j=1}^N\left((\theta_j - c)\overline{\rho}_j(\theta_j) - \int_{\underline{\theta}_j}^{\theta_j}\overline{\rho}_j(\theta)d\theta\right)\right] \\
&= \sum_{i=1}^N a_i
\end{aligned}$$

Therefore, in order for the transfer scheme to satisfy ex ante budget balancedness, $\sum_{i=1}^N a_i \geq 0$.

In order for the transfer scheme to satisfy incentive compatibility, let's first compute the expected utility under truthful reporting:

$$\begin{aligned}
U_i(\theta_i) &= \theta_i\overline{\rho}_i(\theta_i) - \overline{\xi}_i(\theta_i) \\
&= \theta_i\overline{\rho}_i(\theta_i) - a_i - \theta_i\overline{\rho}_i(\theta_i) + \int_{\underline{\theta}_i}^{\theta_i}\overline{\rho}_i(\theta)d\theta + \frac{1}{n-1}\mathbb{E}_{-i}\left[\sum_{j\neq i}\left((\theta_j - c)\overline{\rho}_j(\theta_j) - \int_{\underline{\theta}_j}^{\theta_j}\overline{\rho}_j(\theta)d\theta\right)\right]
\end{aligned}$$

If we integrate the last term in the second equality (e.g. using the integration by parts argument developed earlier) we can write:

$$U_i(\theta_i) = -a_i + \int_{\underline{\theta}_i}^{\theta_i}\overline{\rho}_i(\theta)d\theta + \underbrace{\frac{1}{n-1}\mathbb{E}_{-i}\left[\sum_{j\neq i}\overline{\rho}_j(\theta_j)\left(\theta_j - \frac{1-F_i(\theta_i)}{f_i(\theta_i)} - c\right)\right]}_{\equiv\,\gamma_i}$$

Notice that I have defined the last RHS term as γ_i and that this term does not depend on agent i's type, θ_i. From our previous characterization theorem for linear environments, the mechanism is incentive compatible if and only if

$$U_i(\theta_i) = U_i(\underline{\theta}_i) + \int_{\underline{\theta}_i}^{\theta_i} \overline{\rho}_i(\theta) d\theta$$

Accordingly, we now have a new restriction on the value of a_i:

$$U_i(\theta_i) = U_i(\underline{\theta}_i) + \int_{\underline{\theta}_i}^{\theta_i} \overline{\rho}_i(\theta) d\theta = -a_i + \int_{\underline{\theta}_i}^{\theta_i} \overline{\rho}_i(\theta) d\theta + \gamma_i \implies a_i = \gamma_i - U_i(\underline{\theta}_i)$$

The two conditions on a_i that we have developed so far imply: $\sum_{i=1}^N a_i = \sum_{i=1}^N \gamma_i - \sum_{i=1}^N U_i(\underline{\theta}_i)$. By the definition, we can write:

$$\sum_{i=1}^N \gamma_i = \mathbb{E}_{-i}\left[\sum_{i=1}^N \overline{\rho}_i(\theta_i)\left(\theta_i - \frac{1-F_i(\theta_i)}{f_i(\theta_i)} - c\right)\right]$$

By assumption ($\star$), we know $\sum_{i=1}^N \gamma_i \geq 0$. Therefore:

$$\sum_{i=1}^N \gamma_i \geq 0 \quad \implies \quad \sum_{i=1}^N \gamma_i = \sum_{i=1}^N a_i + \sum_{i=1}^N U_i(\underline{\theta}_i) \geq 0 \quad \iff \quad \sum_{i=1}^N \gamma_i \geq -\sum_{i=1}^N U_i(\underline{\theta}_i)$$

Hence ex ante budget balancedness is satisfied under these two restrictions since:

$$\sum_{i=1}^N a_i = \sum_{i=1}^N \gamma_i - \sum_{i=1}^N U_i(\underline{\theta}_i) \geq 0$$

To show that the restrictions on a_i are sufficient to satisfy incentive compatibility and voluntary participation, I want to construct a_i so that $U_i(\underline{\theta}_i) = 0$ for all $i < N$. This requires that $a_i = \gamma_i$ for all $i < N$, which will be incentive compatible if in fact $U_i(\underline{\theta}_i) = 0$, which it is:

$$U_i(\underline{\theta}_i) \;=\; -a_i + \int_{\underline{\theta}_i}^{\underline{\theta}_i} \overline{\rho}_i(\theta) d\theta + \gamma_i \;=\; -\gamma_i + 0 + \gamma_i \;=\; 0$$

This specification of a_i additionally satisfies voluntary participation for all $i < N$ because $\overline{\rho}_i(\cdot)$ is a nondecreasing function and is bounded below by a nonnegative value by assumption (e.g. $\overline{\rho}_i(\theta) \in [0,1]$ for all $\theta \in \Theta$ and i):

$$U_i(\theta_i) = \underbrace{U_i(\underline{\theta}_i)}_{=\,0} + \int_{\underline{\theta}_i}^{\theta_i} \overline{\rho}_i(\theta) d\theta \geq 0$$

Under this specification, in order for a_i to satisfy budget balancedness we will let $a_N = -\sum_{i<N} a_i$. Accordingly $\sum_{i=1}^N a_i = \sum_{i<N} \gamma_i + (-\sum_{i<N} \gamma_i) = 0$ satisfies budget balancedness. Additionally, the mechanism for agent N satisfies incentive compatibility and voluntary participation:

$$U_i(\theta_i) = U_i(\underline{\theta}_i) + \int_{\underline{\theta}_i}^{\theta_i} \overline{\rho}_i(\theta) d\theta$$

$$= \Big\{ - a_N + 0 + \gamma_N \Big\} + \int_{\underline{\theta}_i}^{\theta_i} \overline{\rho}_i(\theta) d\theta$$

$$= \underbrace{\sum_{i=1}^{N} \gamma_i}_{\geq 0} + \underbrace{\int_{\underline{\theta}_i}^{\theta_i} \overline{\rho}_i(\theta) d\theta}_{\geq 0}$$

where the first term in the last equality is nonnegative by assumption ($\star$) and the second term is nonnegative by the assumption on $\overline{\rho}_i(\cdot)$ being an increasing function. Thus a_N satisfies incentive compatibility and voluntary participation.

Therefore we have shown that there exists a transfer scheme that satisfies incentive compatibility, interim voluntary participation, and ex ante budget balancedness. □

Alternate Sufficiency Proof. Rob (1989) proposes a different transfer scheme that satisfies incentive compatibility, interim voluntary participation and ex ante budget balancedness. Consider the following construction:

$$\xi_i(\theta) = \theta_i \rho(\theta) - \int_{\underline{\theta}_i}^{\theta_i} \rho(\tilde{\theta}, \theta_{-i}) d\tilde{\theta}$$

Substituting the transfer scheme into agent i's expected payoff function we obtain:

$$U_i(\theta_i) = \theta_i \overline{\rho}_i(\theta_i) - \left\{ \theta_i \overline{\rho}_i(\theta_i) - \int_{\underline{\theta}_i}^{\theta_i} \overline{\rho}_i(\tilde{\theta}) d\tilde{\theta} \right\}$$

$$= \int_{\underline{\theta}_i}^{\theta_i} \overline{\rho}_i(\tilde{\theta}) d\tilde{\theta}$$

Now we can show that $U_i(\underline{\theta}_i) = 0$, and therefore the second inequality proves the transfer scheme is incentive compatible by Theorem 1.2.1:

$$U_i(\underline{\theta_i}) = \int_{\underline{\theta}_i}^{\underline{\theta}_i} \overline{\rho}_i(\tilde{\theta}) d\tilde{\theta} = 0$$

Furthermore because we supposed that $\overline{\rho}_i(\cdot)$ is a nondecreasing function and since each agent i and for all $\theta_i \in \Theta_i$ we have $\overline{\rho}_i(\theta_i) \in [0, 1]$, we know that expected utility is bounded below by a nonnegative value and therefore the transfer scheme can be implemented under voluntary participation.

Lastly we must show that the transfer scheme satisfies ex ante budget balancedness.

$$\mathbb{E}_\theta \left[\left(\sum_{i=1}^{N} \xi_i(\theta) - cN \right) \rho(\theta) \right] = \mathbb{E}_\theta \left[\sum_{i=1}^{N} (\theta_i - c) \rho_i(\theta) - \int_{\underline{\theta}_i}^{\theta_i} \rho_i(\tilde{\theta}, \theta_{-i}) d\tilde{\theta} \right]$$

$$= \mathbb{E}_i \left[\sum_{i=1}^{N} (\theta_i - c) \overline{\rho}_i(\theta_i) - \int_{\underline{\theta}_i}^{\theta_i} \overline{\rho}_i(\tilde{\theta}) d\tilde{\theta} \right]$$

$$= \mathbb{E}_i \left[\sum_{i=1}^{N} (\theta_i - c)\overline{\rho}_i(\theta_i) - \overline{\rho}_i(\theta_i) \frac{1 - F_i(\theta_i)}{f_i(\theta_i)} \right]$$

$$= \mathbb{E}_\theta \left[\left(\sum_{i=1}^{N} \left\{ \theta_i - \frac{1 - F_i(\theta_i)}{f_i(\theta_i)} \right\} - cN \right) \rho(\theta) \right] \geq 0$$

where the last line is greater than zero by assumpion ($\star$). Therefore the proposed transfer scheme satisfies ex ante budget balancedness, incentive compatibility and interim voluntary participation. □

Note that the function over which we take expectations in ($\star$) is called **virtual utility**. Importantly, the term $(1 - F_i(\theta_i))/f_i(\theta_i)$ in ($\star$) effectively decreases the agents' valuations and therefore quantifies the inefficiency that arises due to private information. This term is the inverse of the hazard rate: the probability the agent actually values the public good *more* than he reports, conditional on the agent reporting he values the public good at θ.

1.2.3 Myerson and Satterthwaite (JET 1983): Impossibility Theorem

Using a similar characterization to ($\star$), Myerson and Satterthwaite (1983) proceed to show that in a bilateral trade setting the presence of private information about valuation of a tradable good necessarily implies that no ex post efficient social choice function is truthfully implementable under voluntary participation. In essence, they show that when $N = 2$, the ex post efficient provision rule violates the necessary condition, ($\star$). I will not prove this result here, but instead refer the reader to Mas-Colell, Whinston and Green (MWG p. 895) for a clear explication. Instead I will show the impossibility of ex post efficient outcomes in the environment of Mailath and Postlewaite (1990).

I will now prove that ex post efficient social choice functions are not truthfully implementable with voluntary participation in this environment. Since $\rho(\theta)$ is common among agents, the ex post efficient provision rule is to provide the good when the social benefits outweigh the cost:

$$\max_{\rho} \left\{ \Big(\sum_{i=1}^{N} \theta_i - C(N) \Big) \rho(\theta) \right\} \quad \Longrightarrow \quad \rho^*(\theta) = \begin{cases} 1 & \text{if } \sum_{i=1}^{N} \theta_i > C(N) \\ 0 & \text{Otherwise} \end{cases}$$

Before going on to the Impossibility Theorem, a useful integration result is as follows.

Lemma 1.2.3
Expected virtual utility for an agent, i, is equal to the lower bound on Θ_i: $\underline{\theta}_i$.

Proof. Integrate virtual utility by parts:

$$\int_{\underline{\theta}_i}^{\overline{\theta}_i} \left(\theta_i - \frac{1 - F_i(\theta_i)}{f_i(\theta_i)} \right) f_i(\theta_i) d\theta_i = \int_{\underline{\theta}_i}^{\overline{\theta}_i} \theta_i f_i(\theta_i) d\theta_i - \Big\{ (1 - F_i(\theta_i))\theta_i \Big\}_{\underline{\theta}_i}^{\overline{\theta}_i} - \int_{\underline{\theta}_i}^{\overline{\theta}_i} \theta_i f_i(\theta_i) d\theta_i = \underline{\theta}_i$$

□

To show that the ex post efficient provision rule is not truthfully implementable with voluntary participation, I will show that ($\star$) is violated under this provision rule. In what follows, I will assume for simplicity that $C(N) \equiv cN$ as I did before. Notice that the crucial assumption of the proof is that when each agent has the lowest valuation of the public good, there will be no ex post provision, e.g. $\sum_i \theta_i < cN$.

Theorem 1.2.4 (Impossibility of Ex Post Efficiency)
Suppose that in addition to the hypotheses of the last theorem being satisfied, $\bigcap_{i=1}^N \Theta_i \neq \emptyset$ holds. Let $c > \underline{\theta}_i$ for all i. Then there exists no ex post efficient social choice provision rule $\rho : \Theta \to [0,1]$ that is truthfully implementable under voluntary participation.

Proof. Suppose not. Suppose for contradiction that the ex post efficient provision rule satisfies is truthfully implementable under voluntary participation. Then from the last theorem, such a social choice function function must satisify ($\star$). I will substitute the ex post efficient provision rule $\rho^*(\theta)$ into ($\star$) and show that the necessary condition ($\star$) is violated by the ex post efficient outcome. Substiting $\rho^*(\theta)$ amounts to taking the expectation of the virtual utility conditional on the aggregate valuation being greater than the provision cost.

$$\mathbb{E}_\theta \left[\left(\sum_{i=1}^N \left\{ \theta_i - \frac{1 - F_i(\theta_i)}{f_i(\theta_i)} \right\} - cN \right) \rho^*(\theta) \right]$$

$$= \mathbb{E}_\theta \left[\sum_{i=1}^N \left\{ \theta_i - \frac{1 - F_i(\theta_i)}{f_i(\theta_i)} \right\} - cN \,\middle|\, \sum_{i=1}^N \theta_i \geq cN \right]$$

$$= \int_{\underline{\theta_1}}^{\overline{\theta_1}} \cdots \int_{\underline{\theta_N}}^{\overline{\theta_N}} \left(\sum_{i=1}^N \left\{ \theta_i - \frac{1 - F_i(\theta_i)}{f_i(\theta_i)} \right\} - cN \right) \frac{\prod_{i=1}^N f_i(\theta_i)}{Prob\left(\sum_{i=1}^N \theta_i \geq cN\right)} d\theta_1 \cdots d\theta_N$$

$$= \frac{1}{Prob\left(\sum_{i=1}^N \theta_i \geq cN\right)} \sum_{i=1}^N \int_{\underline{\theta_i}}^{\overline{\theta_i}} \left(\theta_i - \frac{1 - F_i(\theta_i)}{f_i(\theta_i)} - c \right) f_i(\theta_i) d\theta_i$$

$$= \frac{1}{Prob\left(\sum_{i=1}^N \theta_i \geq cN\right)} \left(\sum_{i=1}^N \underline{\theta}_i - cN \right)$$

where the last line follows from lemma 1.2.3. By assumption we know that $c > \underline{\theta}_i$ for all i and therefore $\sum_{i=1}^N \theta_i < cN$. Hence we have arrived at the desired contradiction. □

1.2.4 Mailath and Postlewaite (RES 1990): Asymptotic Inefficiency

Having delved into the previous heroic algebra, we can now consider the ex post efficiency of the economy as $N \to \infty$. Mailath and Postlewaite show that as the number of agents increases, the probability of provision converges to zero almost surely.

The logic of the proof is as follows. First identify an optimal provision rule $\rho^n(\theta;\alpha)$ for each n. Show that there exists $\alpha^*(n)$ such that ($\star$) is satisfied with equality and the probability of provision is maximized. Second show that $\alpha^*(n)$ is necessary for optimality because the probability that valuations are above the cost of provision tend to zero as the economy becomes large. Third, show that the expected optimal provision rule tends to zero as the economy gets large. This requires characterizing the behavior of $\alpha^*(n)$ and showing it tends to zero.

Theorem 1.2.5 (Asymptotic Inefficiency)
Suppose that $\{C(n), F^n\}$ defines a sequence of economies where $F^n = (F_1^n, \ldots, F_n^n)$ such that

(i) $\theta_i - \dfrac{1 - F_i^n(\theta_i)}{f_i^n(\theta_i)}$ is strictly increasing in θ_i

(ii) $\exists K > 0$ such that $f_i^n(\theta_i) > K$ (e.g. f has full support)

(iii) $\overline{\theta}_i^n < \infty$

(iv) $\sum_i \underline{\theta}_i^n < C(n)$

for all i and n. Let $\rho^n(\theta)$ be the maximum probability of provision in economy n. Then $lim_{n\to\infty}\rho^n(\theta) = 0$ for any mechanism in the sequence of mechanisms satisfying incentive compatibility, interim voluntary participation and ex ante budget balancedness.

Proof. Consider mechanisms that maximize the probability of provision according to the following maximization problem

$$\max_{\rho} \int_{\underline{\theta}_1}^{\overline{\theta}_1} \cdots \int_{\underline{\theta}_n}^{\overline{\theta}_n} \rho^n(\theta) \prod_{k=1}^{n} f_k^n(\theta_k) d\theta_k$$

$$\text{s.t.} \quad \rho_i(\theta_i) \text{ is increasing for all } i$$

$$\mathbb{E}_\theta \left[\left(\textstyle\sum_{i=1}^n \left\{ \theta_i - \dfrac{1 - F_i^n(\theta_i)}{f_i^n(\theta_i)} \right\} - C(N) \right) \rho^n(\theta) \right] \geq 0$$

Define $\beta_i^n(\theta_i) = \theta_i - (1 - F_i^n(\theta_i))/f_i^n(\theta)$ and $c(n) = C(n)/n$. Note that since $\beta_i^n(\theta_i)$ is strictly increasing by (i), then $\rho^n(\theta)$ is strictly increasing. Set up the Lagrangian with multiplier λ:

$$\mathcal{L} = \int_{\underline{\theta}_1}^{\overline{\theta}_1} \cdots \int_{\underline{\theta}_n}^{\overline{\theta}_n} \Big[1 + \lambda\Big(\sum_{i=1}^n \beta_i^n(\theta_i) - nc(n) \Big)\Big] \rho(\theta) \prod_{k=1}^n f_k(\theta_k) d\theta_k$$

$$= \lambda \int_{\underline{\theta}_1}^{\overline{\theta}_1} \cdots \int_{\underline{\theta}_n}^{\overline{\theta}_n} \Big[\sum_{i=1}^n \beta_i^n(\theta_i) - nc(n) + n\alpha \Big] \rho(\theta) \prod_{k=1}^n f_k(\theta_k) d\theta_k$$

where $n\alpha \equiv 1/\lambda$. Define the provision rule as:

$$\rho^n(\theta, \alpha) = \begin{cases} 1 & \text{if } \dfrac{1}{n} \displaystyle\sum_{i=1}^n \beta_i^n(\theta_i) + \alpha \geq c(n) \\ 0 & \text{otherwise} \end{cases}$$

In order for this provision rule to be optimal, the expected virtual utility must be equal to zero (e.g. the constraint must be binding). So we must now characterize the α that defines the *optimal* provision rule: we will show that there exists $\alpha^*(n)$ such that ($\star$) is satisfied with equality and the probability of provision is maximized. To do so, define

$$\eta_n(\theta) = \frac{1}{n}\sum_{i=1}^{n} \beta_i^n(\theta_i) \qquad \text{and} \qquad G_n(\alpha) = \mathbb{E}_\theta\Big[\Big(\eta_n(\theta) - c(n)\Big)\rho^n(\theta,\alpha)\Big]$$

Now suppose that $\alpha > \gamma$ and notice:

$$0 > \eta_n(\theta) - c(n) + \gamma \implies 0 > \eta_n(\theta) - c(n) + \gamma > \eta_n(\theta) - c(n)$$

And therefore the provision rule $\rho^n(\theta,\gamma)$ assigns positive measure to outcomes with negative virtual surplus. Accordingly larger γ correspond to a larger set of $\theta \in \Theta$ which receive positive measure but have negative virtual surplus. Therefore $G_n(\cdot)$ is decreasing.

Now write:

$$G_n(\alpha) - G_n(\gamma) = \mathbb{E}_\theta\Big[\Big(\eta_n(\theta) - c(n)\Big)\Big(\rho^n(\theta,\alpha) - \rho^n(\theta,\gamma)\Big)\Big]$$

Notice that:

$$\left.\begin{array}{l} \rho^n(\theta,\gamma) = 1 \implies \rho^n(\theta,\alpha) = 1 \\ \rho^n(\theta,\alpha) = 0 \implies \rho^n(\theta,\gamma) = 0 \end{array}\right\} \implies \rho^n(\theta,\alpha) - \rho^n(\theta,\gamma) \geq 0$$

And define the set of $\theta \in \Theta$ such that $\rho^n(\theta,\alpha) > \rho^n(\theta,\gamma)$ as:

$$\begin{aligned} D_n(\alpha,\gamma) &= \Big\{\theta \;\Big|\; \rho^n(\theta,\alpha) \neq \rho^n(\theta,\gamma)\Big\} \\ &= \Big\{\theta \;\Big|\; \eta_n(\theta) + \alpha \geq c(n) \quad \text{and} \quad \eta_n(\theta) + \gamma < c(n)\Big\} \\ &= \Big\{\theta \;\Big|\; \gamma < c(n) - \eta_n(\theta) \leq \alpha\Big\} \end{aligned}$$

Furthermore, notice that:

$$\Big|\alpha - \gamma\Big| \to 0 \implies \rho^n(\theta,\alpha) - \rho^n(\theta,\gamma) \to 0 \implies G_n(\alpha) - G_n(\gamma) \to 0$$

Therefore, since the measure of the set $D_n(\alpha,\gamma)$ will approach zero, $G_n(\cdot)$ is continuous.

Since (1) $\sum_{i=1}^n \overline{\theta}_i > C(n)$ by assumption, (2) we assume (ii) holds, and (3) $1 - F_i^n(\overline{\theta}_i) = 0$, there exists $\theta \in \Theta$ such that $\eta_n(\theta) > 0$. Therefore $G_n(0) > 0$.

Lastly, define $\alpha'(n) = c(n) - \eta_n(\underline{\theta})$. Notice that $\rho^n(\theta,\alpha'(n)) = 1$ for all $\theta \in \Theta$ since we always have $\eta_n(\theta) > \eta_n(\underline{\theta})$. Therefore, by the above lemma (1.2.3) we write:

$$G_n(\alpha'(n)) = \frac{1}{n}\sum_{i=1}^{n} \underline{\theta}_i - c(n) < 0$$

Now we can apply the Intermediate Value Theorem: $\exists\, a^*(n) \in [0, \alpha'(n)]$ such that $G_n(a^*(n)) = 0$. Thus the optimal provision rule is $\rho^n(\theta,\alpha^*(n))$ and the maximum probability of provision is given by $r(n) = \mathbb{E}[\rho^n(\theta,\alpha^*(n))]$. Next we will show that $r(n) \to 0$.

By Chebychev's Inequality,

$$
\begin{aligned}
Pr\Big[\eta_n(\theta) \geq c(n)\Big] &= Pr\left[\eta_n(\theta) - \frac{1}{n}\sum_{i=1}^{n}\underline{\theta}_i \geq c(n) - \frac{1}{n}\sum_{i=1}^{n}\underline{\theta}_i\right] \\
&\leq \frac{\sigma^2}{n\Big(c(n) - \frac{1}{n}\sum_{i=1}^{n}\underline{\theta}_i\Big)^2} \\
&\leq \frac{\sigma^2}{n\varepsilon^2} \to 0 \qquad \text{as } n \to \infty
\end{aligned}
$$

where σ^2 is the upper bound on the variances of $\beta_i^n(\theta)$ which are finite by assumptions (ii) and (iii). Furthermore, by assumption $C(n) > \sum_{i=1}^{n}\underline{\theta}_i$, and therefore $C(n) - \sum_{i=1}^{n}\underline{\theta}_i > \varepsilon$ gives the final inequality.

Therefore, if $\alpha^*(n) \to 0$ as $n \to \infty$, then Chebychev's Inequality again gives us:

$$
\begin{aligned}
Pr\Big[\eta_n(\theta) + \alpha^*(n) \geq c(n)\Big] &= Pr\left[\eta_n(\theta) - \frac{1}{n}\sum_{i=1}^{n}\underline{\theta}_i \geq c(n) - \frac{1}{n}\sum_{i=1}^{n}\underline{\theta}_i - \alpha^*(n)\right] \\
&\leq \frac{\sigma^2}{n\Big(c(n) - \frac{1}{n}\sum_{i=1}^{n}\underline{\theta}_i - \alpha^*(n)\Big)^2} \\
&\leq \frac{\sigma^2}{n(\varepsilon - \alpha^*(n))^2} \to 0 \qquad \text{as } n \to \infty
\end{aligned}
$$

Lastly, Mailath and Postlewaite show that when $\alpha^*(n) \to \hat{a} \neq 0$, $r(n) \to 0$ because we can partition the set

$$
\begin{aligned}
B(n) &= \{\theta \mid \eta(\theta) - c(n) \geq -\alpha^*(n)\} \\
&= D(n) \qquad + \qquad A(n,m) \qquad + \qquad A'(n,m) \\
&= \Big\{\theta \Big| \eta(\theta) \geq c(n)\Big\} + \Big\{\theta \Big| 0 > \eta(\theta) - c(n) > -\frac{\alpha^*(n)}{m}\Big\} + \Big\{\theta \Big| -\frac{\alpha^*(n)}{m} \geq \eta(\theta) - c(n) \geq -\alpha^*(n)\Big\}
\end{aligned}
$$

and write:

$$
\begin{aligned}
G_n\Big(\alpha^*(n)\Big) &= G_n\Big(\alpha^*(n)|\theta \in D(n)\Big) + G_n\Big(\alpha^*(n)|\theta \in A(n.m)\Big) + G_n\Big(\alpha^*(n)|\theta \in A'(n,m)\Big) \\
&= G_n(0) + \int \cdots \int_{A(n,m)\cup A'(n,m)} \Big(\eta_n(\theta) - c(n)\Big) \prod_{k=1}^{n} f_k(\theta_k)d\theta_k
\end{aligned}
$$

Thus to obtain the result we must show that each partition has zero measure in the limit as $n \to 0$. I leave this part of the proof alone for now. □

1.3 Externalities and Social Choice: Rob (JET 1989)

Rob (1989) analyzes the classic problem of bargaining over compensation for a negative externality due to pollution. In this environment, a firm wishes to establish a factory and produce a good that generates a negative externality for each of $i = 1, \dots, n$ agents (residents). Since each agent has veto power over the firm's production decision, the firm must set up a contract that maximizes its profits while sufficiently compensating each agent. However, formulating a profit maximizing compensation is problematic since each agent's losses due to the negative externality are private information.

Suppose that the firm receives a revenue of R if it produces, but then each agent suffers a loss of $\theta_i \in \Theta_i \equiv [\underline{\theta}_i, \overline{\theta}_i]$ drawn from the CDF $F_i(\cdot)$ with accompanying density $f_i(\cdot)$. The distribution has full support over Θ_i for each i. Furthermore, $0 \leq \underline{\theta}_i < \overline{\theta}_i < \infty$ for each i. The value of the loss θ_i is private information for each agent.

In the full information optimum, the Pareto efficient production decision of the firm would be to produce if and only if firm revenues exceed the social cost of the externality:

$$\sum_{i=1}^{n} \theta_i \leq R$$

But due to private information over loss values, the firm must act as a Stackelberg leader in which it first announces a mechanism and then agents report their losses. If reported losses are sufficiently small the firm will produce. Accordingly, a **mechanism** in this environment consists of agents' reporting strategies $(A_i)_{i=1}^n$ such that $A_i = \Theta_i$ for all i, the probability the firm produces $\rho : \Theta \to [0, 1]$, and a compensation scheme $(c_i)_{i=1}^n$ such that $c : \Theta \to \mathbb{R}^N$.

For notational convenience, define the following:

$$\begin{aligned}
\overline{\rho}_i(\theta_i) &= \mathbb{E}_{-i}[\rho(\theta_i, \theta_{-i})] \\
\overline{c}_i(\theta_i) &= \mathbb{E}_{-i}[c(\theta_i, \theta_{-i})] \\
U_i(\theta_i) &= \mathbb{E}_{-i}\Big[\rho(\theta)\Big(c_i(\theta) - \theta_i\Big)\Big] \;\; = \overline{c}_i(\theta_i) - \theta_i \overline{\rho}_i(\theta_i)
\end{aligned}$$

As with the notation before we could define $w(\theta, \widehat{\theta})$ as the utility from misreporting your true type. Then we could also consider $U_i(\theta_i) = w(\theta_i, \theta_i)$.

Given the Stackelberg game described above, Rob considers implementation under incentive compatibility and interim voluntary participation:

$$\begin{aligned}
&(IC) \qquad U_i(\theta_i) \geq \overline{c}_i(\widehat{\theta}_i) - \theta_i \overline{\rho}_i(\widehat{\theta}_i) \qquad \forall\ \theta_i, \widehat{\theta}_i \in \Theta_i \\
&(VP) \qquad U_i(\theta_i) \geq 0 \qquad \forall\ \theta_i \in \Theta_i
\end{aligned}$$

Incentive compatibility is standard in this environment, but voluntary participation requires some intuition. Since agents each have veto power over the firm's production, they can decide to opt out of the firm's contract. If they decide to veto firm production then the state of the world remains the status quo - this is no firm production, no compensation scheme, and no externality generated. Therefore, in order to implement a mechanism with voluntary participation, the firm must make agents better off than the status quo (in expectations).

The following theorem gives necessary and sufficient conditions for a mechanism to be incentive compatible under voluntary participation in this environment.

Theorem 1.3.1 (Characterization with Externalities)
A mechanism $\{(A_i)_{i=1}^n, \rho, c\}$ is incentive compatible and satisfies voluntary participation if and only if

(a) $\rho(\theta) \in [0,1]$ for all $\theta \in \Theta$

(b) $U_i(\overline{\theta}_i) \geq 0$ for all $i = 1, \dots, n$

(c) $\overline{\rho}_i(\cdot)$ is monotonically decreasing

(d) $U_i(\theta_i) = U_i(\overline{\theta}_i) + \displaystyle\int_{\theta_i}^{\overline{\theta}_i} \overline{\rho}_i(\tilde{\theta})d\tilde{\theta}$

Proof. See Theorem (1.2.1). Since the expected payoff function now has a negative in front of $\theta\overline{\rho}_i(\theta_i)$, conditions (c) and (d) are slightly modified from before. Condition (d) ensures incentive compatibility. Condition (b) and the fact that $\overline{\rho}_i(\theta_i)$ is nonnegative, ensures that $U_i(\theta_i) \geq 0$ in condition (d). Thus the mechanism also satisfies voluntary participation. □

Let's now write the firm's profit maximization problem:

$$\max_{\rho,c} \int_{\underline{\theta}_1}^{\overline{\theta}_1} \cdots \int_{\underline{\theta}_n}^{\overline{\theta}_n} \left(\rho(\theta)R - \sum_{i=1}^n c_i(\theta)\right) \prod_{k=1}^n f_k(\theta_k)d\theta_k$$

$$\text{s.t.} \quad U_i(\theta_i) \geq \overline{c}_i(\widehat{\theta}_i) - \theta_i\overline{\rho}_i(\widehat{\theta}_i) \qquad \forall\ \theta_i, \widehat{\theta}_i \in \Theta_i \quad (IC)$$

$$U_i(\theta_i) \geq 0 \qquad \forall\ \theta_i \in \Theta_i \qquad (VP)$$

The next result is similar to Theorem (1.2.2).

Theorem 1.3.2 (Characterization of Profit Maximizing Decision)
Suppose $\rho : \Theta \to [0,1]$ is in the *argmax* of the following:

$$\max_{\rho} \mathbb{E}_\theta \left[\left(R - \sum_{i=1}^n \left\{\theta_i + \frac{F_i(\theta_i)}{f_i(\theta_i)}\right\}\right)\rho(\theta)\right] - \sum_{i=1}^n U_i(\overline{\theta}_i) \tag{2}$$

$$\text{s.t.} \quad \overline{\rho}_i(\theta_i) = \mathbb{E}_{-i}\Big[\rho(\theta_i, \theta_{-i})\Big] \text{ is monotonically decreasing}$$

Also suppose that the compensation scheme for each agent $i = 1, \dots, n$ is a function as follows:

$$c_i(\theta) = \theta_i\rho(\theta) + \int_{\theta_i}^{\overline{\theta}_i} \rho(\theta, \theta_{-i})d\theta$$

Then the mechanism $(\rho, \{c_i\}_{i=1}^n)$ maximizes the firm's expected profit.

Proof. The objective function (2) is the analogous necessary condition for a mechanism to be incentive compatible under voluntary participation from Theorem (1.2.2). Accordingly, any mechanism that maximizes (2) will satisfy the constraint set of the firm's maximization problem. Therefore we must only show that the optimal mechanism $\rho(\theta)$ satisfying (2) is in the *argmax* of the firm's maximization problem. To show this, first note that by integrating by parts we obtain:

$$\begin{aligned}\mathbb{E}_i\left[\int_{\theta_i}^{\bar{\theta}_i}\bar{\rho}_i(x)dx\right] &= \int_{\underline{\theta}_i}^{\bar{\theta}_i}\bar{\rho}_i(x)F_i(x)dx \\ &= \mathbb{E}_i\left[\bar{\rho}_i(\theta_i)\frac{F_i(\theta_i)}{f_i(\theta_i)}\right]\end{aligned}$$

We will now substitute the compensation scheme into the firm's expected profit function:

$$\begin{aligned}\mathbb{E}_\theta\left[\rho(\theta)R-\sum_{i=1}^n c_i(\theta)\right] &= \mathbb{E}_i\left[\bar{\rho}_i(\theta_i)R-\sum_{i=1}^n\left\{\theta_i\bar{\rho}_i(\theta_i)+\int_{\theta_i}^{\bar{\theta}_i}\bar{\rho}_i(x)dx\right\}\right] \\ &= \mathbb{E}_i\left[\left(R-\sum_{i=1}^n\theta_i\right)\bar{\rho}_i(\theta_i)\right]-\sum_{i=1}^n\mathbb{E}_i\left[\int_{\theta_i}^{\bar{\theta}_i}\bar{\rho}_i(x)dx\right] \\ &= \mathbb{E}_i\left[\left(R-\sum_{i=1}^n\left\{\theta_i+\frac{F_i(\theta_i)}{f_i(\theta_i)}\right\}\right)\bar{\rho}_i(\theta_i)\right] \\ &= \mathbb{E}_\theta\left[\left(R-\sum_{i=1}^n\left\{\theta_i+\frac{F_i(\theta_i)}{f_i(\theta_i)}\right\}\right)\rho(\theta)\right]\end{aligned}$$

Since the firm's expected profit function is equivalent to the objective function (2) up to a constant (e.g. $\sum_i U_i$), any mechanism $\rho(\theta)$ that satisfies one must satisfy the other. $\square$

Theorem 1.3.3 (Optimal Mechanism)
If $\theta_i+F_i(\theta_i)/f_i(\theta_i)$ is monotonically increasing for all i, then the following constitutes a profit maximizing mechanism:

$$\rho^*(\theta)=\begin{cases}1 & \text{if } \displaystyle\sum_{i=1}^n\left\{\theta_i+\frac{F_i(\theta_i)}{f_i(\theta_i)}\right\}\le R \\ 0 & \text{otherwise}\end{cases}$$

$$c_i^*(\theta)=\theta_i\rho^*(\theta)+\int_{\theta_i}^{\bar{\theta}_i}\rho^*(\tilde{\theta},\theta_{-i})d\tilde{\theta}$$

Proof. In order for the proposed mechanism to be profit maximizing, the hypotheses of Theorems 1.3.1 and 1.3.2 must be satisfied under implementation of that mechanism. This proof is analogous to the proof of the sufficient condition of Theorem 1.2.2. Condition (a) is trivially satisfied. Condition (c) is satisfied by the assumption that $\theta_i+F_i(\theta_i)/f_i(\theta_i)$ is monotonically increasing.

We know that condition (d) is satisfied if and only if the mechanism induces truthtelling. Accordingly conditions (b) and (d) are satisfied simultaneously. To see this, substitute the production decision and compensation scheme, ρ^* and c^*, into agent i's expected payoff:

$$\begin{aligned} U_i(\theta_i|\rho^*, c^*) &= \mathbb{E}_{-i}\Big[c_i^*(\theta) - \theta_i \rho^*(\theta)\Big] \\ &= \mathbb{E}_{-i}\left[\int_{\theta_i}^{\overline{\theta}_i} \rho^*(\theta, \theta_{-i}) d\theta\right] \end{aligned}$$

Therefore, condition (d) is satisfied if and only if $U_i(\overline{\theta}_i|\rho^*, c^*) = 0$, which simultaneously satisfies condition (b). □

It remains to show that this economy exhibits asymptotic inefficiency. Notice that the first three hypothesis of Theorem 1.2.5 are satisfied in this evironment, albeit in tailored forms:

(i) $\theta_i + \dfrac{F_i(\theta_i)}{f_i(\theta_i)}$ is strictly increasing in θ_i

(ii) $\exists K > 0$ such that $f_i(\theta_i) > K$ (e.g. f has full support)

(iii) $\overline{\theta}_i < \infty$

Rob adds an environment specific assumption (A1) that acts as the remaining hypothesis: $\underline{\theta} < r < \overline{\theta}$ where all agents have identical $\Theta \equiv [\underline{\theta}, \overline{\theta}]$ and distributions F. We define r as the firm's profit per agent. Accordingly, define $R_n \equiv r \times n$, which is the firm's profit in an n-agent economy. The assumption ensures that the firm's problem is non-trivial in much the same way hypothesis (iv) did.

Rob also adds a technical assumption so that he can apply the Central Limit Theorem in the following theorem, call it (A2):

$$\frac{\mathbb{E}_\theta[\theta] - \underline{\theta}}{Var_\theta[\theta]} < \frac{\overline{\theta} - \underline{\theta}}{Var_\theta[\theta + z(\theta)]}$$

where $z(\theta) = F(\theta)/f(\theta)$, and the expectations and variances have support over all agents, $\Theta \equiv \Theta \times \cdots \times \Theta$.

Theorem 1.3.4 (Asymptotic Inefficiency With An Externality)
Under the assumptions above, as $n \to \infty$

$$Pr\left(\sum_{i=1}^{n}\left\{\theta_i + \frac{F(\theta_i)}{f(\theta_i)}\right\} \leq R_n \;\middle|\; \sum_{i=1}^{n}\theta_i \leq R_n\right) \to 0$$

Proof. Given that we defined above $z_i = F(\theta_i)/f(\theta_i)$ write:

$$Pr\left(\sum_{i=1}^{n}\left\{\theta_i + \frac{F(\theta_i)}{f(\theta_i)}\right\} \leq R_n \;\middle|\; \sum_{i=1}^{n}\theta_i \leq R_n\right) = \frac{Pr\left(\sum_{i=1}^{n}\{\theta_i + z_i\} \leq R_n\right)}{Pr\left(\sum_{i=1}^{n}\theta_i \leq R_n\right)} \equiv \frac{A_n}{B_n}$$

Apply the central limit theorem to A_n and B_n:

$$\begin{aligned}
A_n &= Pr\left(\sum_{i=1}^{n}\{\theta_i + z_i\} \leq R_n\right) \\
&= Pr\left(\frac{\sum_{i=1}^{n}\{\theta_i + z_i\} - n\overline{\theta}}{n^{1/2}Var(\theta+z)^{1/2}} \leq \frac{nr - n\overline{\theta}}{n^{1/2}Var(\theta+z)^{1/2}}\right) \\
&\approx \phi\left(n^{1/2}\cdot\frac{r-\mathbb{E}[\theta]}{Var(\theta)}\right) \\
&\equiv \phi(n^{1/2}\ a_r) \\
B_n &= Pr\left(\sum_{i=1}^{n}\theta_i \leq R_n\right) \\
&= Pr\left(\frac{\sum_{i=1}^{n}\theta_i - n\mathbb{E}(\theta)}{n^{1/2}Var(\theta)^{1/2}} \leq \frac{nr - n\mathbb{E}(\theta)}{Var(\theta)^{1/2}}\right) \\
&\approx \phi\left(n^{1/2}\cdot\frac{r-\mathbb{E}(\theta)}{Var(\theta)^{1/2}}\right) \\
&\equiv \phi(n^{1/2}\ b_r)
\end{aligned}$$

Thus:

$$\lim_{n\to\infty}\frac{A_n}{B_n} = \lim_{n\to\infty}\frac{\phi(n^{1/2}\ a_r)}{\phi(n^{1/2}\ b_r)}$$

If $b_r \geq 0$ then $A_n \to 0$ and $B_n \to B \neq 0$. Therefore the probability of production approaches zero as the number agents becomes large. If $b_r < 0$ then both A_n and B_n approach zero in the limit. But by L'Hospital's rule A_n approaches zero faster. □

1.4 Synthesis: Chari and Jones (ET 2000)

Chari and Jones (ET 2000) consider environments with global externalities, such as pollution and public goods provision. They show how distortions due to free-rider problems in global externality environments are equivalent to distortions that arise in environments with complementary monopolies. This equivalence challenges Coase's argument that properly defined property rights enables competitive outcomes to be efficient. In fact, Chari and Jones show that greater competition generates inefficiency due to complementarities. In this section I will go through their illustrative examples of the free-rider / complementarity monopoly equivalence.

Chari and Jones consider an environment in which there are n residents of a town. Each resident owns an equally sized plot of land and has utility over a numeraire good m, and has disutility from smoke that may billow over their property: $u_i(m, s) = m - (s/n)$. There is a private good, q, that is consumed by an outside town. Assume that demand for the private good is linear:

$$D(p) = a - bp$$

Firms are perfectly competitive. Consider a stand-in firm that produces the private good, and for each unit produced the firm generates smoke as a byproduct. Smoke will disseminate uniformly over all residents' land. The firm requires no inputs to production.

1.4.1 Competitive versus Lindahl Equilibrium

A competitive equilibrium consists of residents' allocation $\{m_i\}_{i=1}^n$, the firm's allocation $\{q\}$ and prices $\{p\}$ such that

1. Given prices, endowment (e), and smoke, residents choose consumption m_i to maximize utility:

$$\max_{m_i} m_i - \frac{s}{n}$$
$$\text{s.t.} \quad m_i \leq e$$

2. Given prices, firms choose output q to maximize profits:

$$\max_{q} pq$$
$$\text{s.t.} \quad q = s$$

3. Markets clear: $q^* = D(p^*)$

Since firms face a given price and demand, firms produce where price equals marginal cost: $p^* = 0$. By market clearing, then, equilibrium production is $q^* = a - bp^*$. Lastly, residents will consume their entire endowment, $m_i^* = e$ for all i.

Next consider a Lindahl equilibrium, in which households make decisions as if they consumed a personalized quantity of the externality commodity (smoke). Clearly this is an abstraction since the externality is purely public by construction of the environment: smoke billows uniformly over all of town. The abstraction affords us knowledge of how agents would behave if they could internalize the externality.

A Lindahl equilibrium consists of residents' allocation $\{m_i, s_i\}_{i=1}^n$, firm allocation $\{q, s_i^f\}_{i=1}^n$ and prices $\{p, p_i\}_{i=1}^n$ such that

1. Given $\{p_i\}_{i=1}^n$, the residents' allocation solves:

$$\max_{m_i, s_i} m_i - \frac{s_i}{n}$$
$$\text{s.t.} \quad m_i + p_i s_i \leq e$$

2. Given prices $\{p, p_i\}_{i=1}^n$, firms choose output q to maximize profits:

$$\max_q \; pq - \sum_{i=1}^n p_i s_i^f$$

$$\text{s.t.} \quad q = s_i^f \quad \forall\, i$$

3. Markets clear: $q^* = D(p^*)$, $s_i = s_i^f$, and $q = s_i$ for all i

Since utility of the residents is linear, the equilibrium price must be given by $p_i^* = 1/n$ for all i. Also by linearity, perfectly competitive firms produce where price is equal to marginal cost: $p = \sum_{i=1}^n p_i$. Since $p_i = 1/n$ for all i, $p^* = n(1/n) = 1$. Then by market clearing we obtain $q^* = s_i^* = a - b$ for all i.

1.4.2 Global Externalities and Complementary Monopolies

Now consider an environment where residents own the property rights over their land. This means that residents do not need to consume smoke. Further consider a market for pollution permits in which residents sell the firm the right to pollute in a competitive spot market. Each agent is given an endowment S of smoke rights they may sell.

A competitive equilibrium consists of residents' allocation $\{m_i, s_i\}_{i=1}^n$, firm allocation $\{q, s_i^f\}_{i=1}^n$ and prices $\{p, p_i\}_{i=1}^n$ such that:

1. Given $\{p_i\}_{i=1}^n$, the residents' allocation solves:

$$\max_{m_i, s_i} \; m_i - \frac{s_i}{n}$$

$$\text{s.t.} \quad m_i \le e + p_i s_i, \qquad s_i \le S$$

2. Given prices $\{p, p_i\}_{i=1}^n$, firms choose output q and pollution permit purchases s_i^f to maximize profits:

$$\max_q \; pq - \sum_{i=1}^n p_i s_i^f$$

$$\text{s.t.} \quad q = s_i^f \quad \forall\, i$$

3. Markets clear: $q = D(p)$, $s_i = s_i^f$, and $q = s_i$ for all i

Optimally, $p_i^* = 1/n$ for all i and $p^* = 1$ as with the Lindahl equilibrium. Then accordingly $q^* = a - b$ as well. Therefore, by endowing residents with property rights to smoke we have achieved the Lindahl equilibrium allocation.

However, Chari and Jones critique the the notion that the perfectly competitive environment is a useful abstraction for understanding the externality problem in the presence of residents' property rights. In particular, they understand externality problems to be plagued by a hold-up problem: residents own the rights to smoke over their land and may single-handedly veto production of the firm. This is clearly suboptimal - if residents allow some pollution then they collect rents, thereby increasing their consumption. Furthermore notice that the rate of disutility for smoke is less than one ($1/n$). In the following let's consider how agents may strategically interact when given property rights to pollution over their land.

Continue to assume that firms are perfectly competitive. Accordingly, firms will continue to choose output and smoke purchases to maximize profits, while given prices that are set by residents'. Market clearing conditions still require that (1) at a price p^* output demanded will be equal to output supplied and (2) $q = s_i$ for all i. We will further assume that residents know the relationship between price and demand from the market clearing conditions.

A symmetric Nash equilibrium consists of residents' pricing strategies and payoffs $\{p_i, u_i\}_{i=1}^n$ that solve:

$$\max_{m_i, s_i, p_i} \; m_i - \frac{s_i}{n}$$

$$\text{s.t.} \quad \begin{aligned} m_i &\leq e + p_i s_i \\ s_i &\leq S \end{aligned}$$

and where residents know $q(p) = a - b\sum_{i=1}^n p_i$ and $q = s_i$ for all i.

Substituting the resident's budget constraint into their objective function, and by substituting market clearing conditions into the objective function we can rewrite the resident's maximization problem as:

$$\max_{p_i} \; \left(p_i - \frac{1}{n}\right) q(p)$$

$$\text{s.t.} \quad s_i \leq S$$

Taking first order conditions, we obtain:

$$\left(a - b\sum_{j=1}^n p_j\right) + \left(p_i - \frac{1}{n}\right)(-b) = 0$$

By impositing symmetry (e.g. $\bar{p} \equiv p_i = p_j$ for all i, j) we can rewrite the FOC as:

$$a - bn\bar{p} = \left(\bar{p} - \frac{1}{n}\right) b$$

$$p_i^* = \frac{1}{n+1}\left(\frac{a}{b} + \frac{1}{n}\right)$$

which gives us the price each symmetric resident charges per unit of smoke produced. Then the price of the private good is $p^* = np_i^*$ for all i. From the market clearing condition, then $q^* = a - bp^*$. Therefore, p^* is less than one and q^* is less than $a - b$: compared to the Lindahl equilibrium allocation, equilibrium output is lower in the environment with strategic price setting.

Assigning property rights to residents produces dramatic inefficiency relative to the Lindahl equilibrium when the number of residents becomes large. To see this take the limit as $n \to \infty$. Then $p^* \to a/b$, $p_i^* \to 0$ and $q^* \to 0$. Equilibrium output is then very low. This is surprising, as increasing competition diminishes eliminate monopoly power. However, in this environment increasing the number of agents actually exacerbates monopoly power, because the hold-up problem hinges on complementary monopolies as inputs to production. This is reminiscent of Diamond and Mirrlees' (1971) intermediate taxation result: introducing distortions on inputs to production is inefficient.

1.4.3 Public Goods and Complementary Monopolies

Chari and Jones test the robustness of the inefficiency by considering a reallocation of property rights that shifts the nature of the global externality. Now consider an independent third party that gains no utility from the consumption of the firm's private good or from pollution. The third party owns the pollution rights to each equally sized plot of land, S, and sells them to residents and firms. Let x_i be resident i's smoke rights purchase and note that utility is increasing in resident smoke rights.

The reassignment of property rights creates a public goods environments, in the sense that agents are better off by allowing some smoke but free riding generates an inefficiently low smoke allowance.

A competitive equilibrium of this environment consists of residents' allocation $\{m_i, x_i\}_{i=1}^n$, the firm's allocation $\{q, x_i^f\}_{i=1}^n$, and prices $\{p, p_i\}_{i=1}^n$ such that:

1. Given prices, residents choose consumption and smokes rights purchases to maximize uility:

$$\max_{m_i, x_i} m_i + \frac{x_i}{n}$$

$$\begin{aligned} \text{s.t.} \quad m_i + p_i x_i &\leq e \\ x_i &\leq S \end{aligned}$$

2. Given prices, the firm chooses production and smoke rights purchases to maximize profit:

$$\max_{q, x_i^f} pq - \sum_{i=1}^n p_i x_i^f$$

$$\begin{aligned} \text{s.t.} \quad q &= x_i^f \; \forall i \\ x_i^f &\leq S \; \forall i \end{aligned}$$

3. Market clearing: $q = D(p)$, $q = x_i^f$, and $S = x_i + x_i^f$ for all i.

When a third party owns the property rights, optimal prices are $p_i^* = 1/n$ and $p^* = \sum_i p_i^* = 1$. Optimal output and firm's smoke right purchases then achieve the Lindahl equilibrium allocation level, $q^* = a - b$. Resident smoke rights are $x_i = S - q^* = S - (a - b)$.

Now consider the same environment when we relax the assumption of perfect competition in the market for smoke rights. Instead consider an environment in which residents can set the price that they are willing to pay for smoke rights. Notice that the price setting environment here is similar to Mailath and Postlewaite's public goods problem in the sense that a friction (here excludability of property rights) increases the cost of provision - here, the provision of property rights.

We will use the same Nash equilibrium concept as in the previous section. A symmetric Nash equilibrium consists of residents' pricing strategies and payoffs $\{p_i, u_i\}_{i=1}^n$ that solve:

$$\max_{m_i, x_i, p_i} \; m_i + \frac{x_i}{n}$$

$$\text{s.t.} \quad \begin{aligned} m_i + p_i x_i &\leq e \\ x_i &\leq S \end{aligned}$$

and where residents know $q(p) = a - b\sum_{i=1}^n p_i$ and $q = x_i$ for all i, and $S = x_i^f + x_i$ for all i.

Substituting the resident's budget constraint into their objective function, and by substituting market clearing conditions into the objective function we can rewrite the resident's maximization problem as:

$$\max_{p_i} \; \left(\frac{1}{n} - p_i\right)(S - q(p))$$

$$\text{s.t.} \quad q(p) \leq S$$

Taking first order conditions, we obtain:

$$-\left(S - \left(a - b\sum_{j=1}^n p_j\right)\right) + \left(\frac{1}{n} - p_i\right) b = 0$$

By impositing symmetry (e.g. $\bar{p} \equiv p_i = p_j$ for all i, j) we can rewrite the FOC as:

$$S - (a - bn\bar{p}) = \left(\frac{1}{n} - \bar{p}\right) b$$

$$p_i^* = \frac{1}{n+1}\left(\frac{a - S}{b} + \frac{1}{n}\right)$$

which gives us the price each symmetric resident charges per smoke right purchased. Then the price of the private good is $p^* = np_i^*$ for all i. From the market clearing condition production is $q^* = a - bp^*$ and residents purchase $x_i^* = S - q^*$ units of smoke rights. In this case we see that as the number of residents gets large, e.g. $n \to \infty$, then prices are $p_i^* \to 0$ and $p^* \to (a - S)/b$, output is $q \to a - (a - S) = S$ and resident smoke rights are $x_i = S - q \to 0$. Therefore we find that when a third party owns and sells the smoke rights, there is maximum pollution. Again this is a dramatic inefficiency relative to the Lindahl equilibrium allocation. Except here, the inefficiency takes the opposite extreme. Hence the complementary monopoly problem manifests itself in different inefficiencies depending on the allocation of property rights in the economy.

This environment is like Mailath and Postlewaite's in the following sense. A third party provides a public good of clean air, while agents pay a contribution $p_i x_i$. Each agent may contribute less to the provision of the public good than their true valuation. Hence the complementary monopoly problem manifests itself in a pricing friction by which the price does not reflect residents' marginal valuation of clean air. Instead, as the population becomes large ($n \to \infty$) no clean air is provided because residents free ride.

1.5 Exercises: Prelim Questions

To be completed.

2 Mirrleesian Economies

In this chapter I will consider the set of tools developed by Mirrlees' (1971), in his seminal contribution to the analysis of mechanism design. I will begin by considering a static environment in which agents might be one of two types. I will use this simple environment to illustrate the fundamental technical issues that arise in this class of economies. I will then enlarge the state space to consist of a large but finite number of types and show how to approach this more technically difficult problem. I will then consider a continuum of types. Lastly I will analyze a dynamic Mirrleesian environment and provide two decentralizations along the lines of Golosov and Tsyvinski's (2004) disability insurance contracts and Kocherlakota's (2003) capital tax.

2.1 Static Economy with Two Types

Consider a static environment with a continuum of agents who gain utility from consumption, c, and disutility from labor, ℓ:

$$U(c, \ell) = u(c) - v(\ell)$$

Assume that $u(\cdot)$ is strictly increasing and strictly concave. Assume that $v(\cdot)$ is strictly increasing and convex. Let $U(c, \ell)$ be twice continuously differentiable and satisfy Inada conditions.

When an agent exerts labor he produces $y = \theta\ell$, where $\theta \in \Theta = \{\theta_L, \theta_H\}$ is the labor productivity of the agent. Let $\theta_H > \theta_L$. We assume that labor hours and productivity are privately known by the agent and therefore output is the only observable component of production. Nature assigns each agent a stochastic, iid productivity draw according to the density function $\pi(\cdot)$. By the law of large numbers, $\pi(\theta)$ is also the fraction of the population that receive a productivity draw of θ. Therefore we are considering an economy with aggregate certainty over the distribution of productivity draws.

A **mechanism** of this environment consists of agents' type-reporting strategies $(A_i)_{i \in [0,1]}$ and outcome functions $c : A \to \mathbb{R}_+$ and $y : A \to \mathbb{R}_+$. Where, as before, A denotes the message space. Accordingly, a **revelation mechanism** has $A_i = \Theta_i$ for all $i \in [0, 1]$ and the outcome functions $c : \Theta \to \mathbb{R}_+$ and $y : \Theta \to \mathbb{R}_+$. Lastly by applying the revelation principle, a Bayesian Nash Equilbrium is implementible under truth-telling if we impose the following incentive compatibility constraints:

$$u\Big(c(\theta_H)\Big) - v\Big(\frac{y(\theta_H)}{\theta_H}\Big) \geq u\Big(c(\theta_L)\Big) - v\Big(\frac{y(\theta_L)}{\theta_H}\Big)$$

$$u\Big(c(\theta_L)\Big) - v\Big(\frac{y(\theta_L)}{\theta_L}\Big) \geq u\Big(c(\theta_H)\Big) - v\Big(\frac{y(\theta_H)}{\theta_L}\Big)$$

We will apply the revelation principle to this environment so that we can restrict our attention to revelation mechanisms and truthfully implement a constrained Pareto efficient allocation. But before analyzing the environment with private information and characterizing the CPE allocation, let's consider the full information optimum. In particular we will characterize a **Pareto efficient allocation**, $\{c(\theta), y(\theta)\}_{\theta\in\Theta}$. The Social Planner's Problem is as follows:

$$\max_{c(\theta),y(\theta)} \sum_{\theta\in\{\theta_L,\theta_H\}} \pi(\theta)\left[u\Big(c(\theta)\Big) - v\Big(\frac{y(\theta)}{\theta}\Big)\right]$$

$$\text{s.t.} \quad \sum_{\theta}\pi(\theta)c(\theta) \leq \sum_{\theta}\pi(\theta)y(\theta) \qquad \text{(RC, } \lambda\text{)}$$

where the resource constraint ensures the that any planner's allocation is resource feasible. Note that I have assumed that the Pareto weights are exactly equal to the measure of each type in the population.

To find the full information optimal allocation take FOCs, letting λ be the multiplier on the resource constraint:

$$u'(c(\theta)) = \lambda \qquad \text{for all } \theta\in\Theta$$

$$\frac{1}{\theta}v'\Big(\frac{y(\theta)}{\theta}\Big) = \lambda \qquad \text{for all } \theta\in\Theta$$

Accordingly the FOCs imply that:

1. Perfect risk sharing is an optimal outcome, $c(\theta_H) = c(\theta_L)$
2. More productive agents will produce more output:

$$\frac{1}{\theta_H}v'\Big(\frac{y(\theta_H)}{\theta_H}\Big) = \frac{1}{\theta_L}v'\Big(\frac{y(\theta_L)}{\theta_L}\Big)$$

$$v'\Big(\frac{y(\theta_H)}{\theta_H}\Big) > v'\Big(\frac{y(\theta_L)}{\theta_L}\Big)$$

$$\frac{y(\theta_H)}{\theta_H} > \frac{y(\theta_L)}{\theta_L}$$

$$y(\theta_H) > y(\theta_L)$$

3. The marginal utility of consumption is equal to the marginal disutility from labor, and hence there is no distortion in the full information optimum:

$$u'(c(\theta)) = \frac{1}{\theta}v'\Big(\frac{y(\theta)}{\theta}\Big) \qquad \text{for all } \theta\in\Theta$$

But notice that the full information optimum will violate the incentive compatibility constraint of a high productivity agent. To show this, suppose not:

$$u\Big(c(\theta_H)\Big) - v\Big(\frac{y(\theta_H)}{\theta_H}\Big) \geq u\Big(c(\theta_L)\Big) - v\Big(\frac{y(\theta_L)}{\theta_H}\Big)$$

$$v\Big(\frac{y(\theta_H)}{\theta_H}\Big) \leq v\Big(\frac{y(\theta_L)}{\theta_H}\Big)$$

$$y(\theta_H) \leq y(\theta_L)$$

The intuition for this result is that if type were private information, then under the full information optimal allocation the high productivity type would prefer to report he is a low productivity agent - because he would consume the same amount but work fewer hours to produce $y(\theta_L)$.

We will now characterize the solution to a social planner's problem with private information. Given that only the high-type's incentive compatibility constraint was violated in the full information optimum, we will guess that that we can drop the low-type's incentive compatibility constraint from the private information planning problem. Thus, the "relaxed problem" is:

$$\max_{c(\theta),y(\theta)} \sum_{\theta} \pi(\theta) \left[u\Big(c(\theta)\Big) - v\Big(\frac{y(\theta)}{\theta}\Big) \right]$$

$$\text{s.t.} \quad u\Big(c(\theta_H)\Big) - v\Big(\frac{y(\theta_H)}{\theta_H}\Big) \geq u\Big(c(\theta_L)\Big) - v\Big(\frac{y(\theta_L)}{\theta_H}\Big) \qquad (\text{ICH}, \pi(\theta_H)\ \mu)$$

$$\sum_{\theta} \pi(\theta) c(\theta) \leq \sum_{\theta} \pi(\theta) y(\theta) \qquad (\text{RC}, \lambda)$$

The first order conditions for this relaxed problem are:

$$(1+\mu)\ u'(c(\theta_H)) = \lambda$$

$$\left(1 - \mu \frac{\pi(\theta_H)}{\pi(\theta_L)}\right)\ u'(c(\theta_L)) = \lambda$$

$$\frac{1}{\theta_H} v'\left(\frac{y(\theta_H)}{\theta_H}\right) = \lambda$$

$$\frac{1}{\theta_L} v'\left(\frac{y(\theta_L)}{\theta_L}\right) - \mu \frac{\pi(\theta_H)}{\pi(\theta_L)} \frac{1}{\theta_H} v'\left(\frac{y(\theta_L)}{\theta_H}\right) = \lambda$$

We can now characterize the constrained optimum with the following five properties.

<u>P1: $c(\theta_H) > c(\theta_L)$</u>

The constrained optimum allocation will assign more consumption to high productivity types. To see this combine the two FOCs for consumption and note that $\mu > 0$:

$$(1+\mu)\ u'(c(\theta_H)) = \left(1 - \frac{\pi(\theta_H)}{\pi(\theta_L)}\right)\ u'(c(\theta_L))$$

$$u'(c(\theta_H)) - u'(c(\theta_L)) = -\mu\ u'(c(\theta_H)) - \frac{\pi(\theta_H)}{\pi(\theta_L)}\ u'(c(\theta_L))\ < 0$$

$$u'(c(\theta_H)) < u'(c(\theta_L))$$

$$c(\theta_H) > c(\theta_L)$$

where the last line follows from the strict concavity of $u(\cdot)$. Hence, in line with our intuition for why the high-type's incentive compatibility constraint was violated under the full information optimum, in order to resolve incentive issues the planner must offer the high-type a carrot in the form of higher consumption.

P2: $y(\theta_H) > y(\theta_L)$

The constrained optimum will require high productivity agents to produce more, to earn their higher consumption. Given that $c(\theta_H) > c(\theta_L)$, we then know $u(c(\theta_H)) > u(c(\theta_L))$. From the incentive compatibility constraint:

$$u\Big(c(\theta_H)\Big) - v\Big(\frac{y(\theta_H)}{\theta_H}\Big) = u\Big(c(\theta_L)\Big) - v\Big(\frac{y(\theta_L)}{\theta_H}\Big)$$

$$v\Big(\frac{y(\theta_H)}{\theta_H}\Big) > v\Big(\frac{y(\theta_L)}{\theta_H}\Big)$$

$$y(\theta_H) > y(\theta_L)$$

where the last line follows from the strict monotonicity of $v(\cdot)$.

P3: No Distortion At The Top

The constrained optimum will not distort high productivity agents' consumption/labor decision away from the full information optimum. This is easily obtained from combining the consumption and labor FOCs for the high type:

$$u'(c(\theta_H)) = \frac{1}{\theta_H} v'\left(\frac{y(\theta_H)}{\theta_H}\right)$$

P4: Distortion At The Bottom

The constrained optimum will distort low productivity agents' consumption/labor decision away from the full information optimum. This is easily obtained from combining the consumption and labor FOCs for the low type:

$$\left(1 - \mu\frac{\pi(\theta_H)}{\pi(\theta_L)}\right) u'(c(\theta_L)) = \frac{1}{\theta_L} v'\left(\frac{y(\theta_L)}{\theta_L}\right) - \mu\frac{\pi(\theta_H)}{\pi(\theta_L)}\frac{1}{\theta_H} v'\left(\frac{y(\theta_L)}{\theta_H}\right)$$

$$u'(c(\theta_L)) - \frac{1}{\theta_L} v'\left(\frac{y(\theta_L)}{\theta_L}\right) = \mu\frac{\pi(\theta_H)}{\pi(\theta_L)}\left[u'(c(\theta_L)) - \frac{1}{\theta_H} v'\left(\frac{y(\theta_L)}{\theta_H}\right)\right]$$

We will simplify the RHS. We can use the information that

$$c(\theta_H) > c(\theta_L) \implies u'(c(\theta_H)) < u'(c(\theta_L))$$

$$y(\theta_H) > y(\theta_L) \implies -v'\left(\frac{y(\theta_H)}{\theta_H}\right) < -v'\left(\frac{y(\theta_L)}{\theta_H}\right)$$

and use the result that there is no distortion for high types, in order to write:

$$u'(c(\theta_L)) - \frac{1}{\theta_H} v'\left(\frac{y(\theta_L)}{\theta_H}\right) > u'(c(\theta_H)) - \frac{1}{\theta_H} v'\left(\frac{y(\theta_H)}{\theta_H}\right) = 0$$

Therefore, since $\mu > 0$ we know that the RHS is positive. Therefore the LHS is positive:

$$u'(c(\theta_L)) > \frac{1}{\theta_L} v'\left(\frac{y(\theta_L)}{\theta_L}\right)$$

which shows that the low-type's decision is distorted away from the full information optimum.

The intuition for P2 and P3 is that since the low-type IC is slack, the planner can decrease the low-type's consumption and increase his production by some small $\varepsilon > 0$ and still satisfy his incentivity compatibility constraint. By doing so, the planner increases the cost of misreporting and therefore makes it a less attractive strategy to high types.

P5: Low-Type IC Is Slack

To check that our initial guess that low productivity agents' incentive compatibility constraint is in fact slack (and hence we were correct to exclude it from the planner's problem), rearrange the low-type's IC as follows:

$$u\left(c(\theta_L)\right) - v\left(\frac{y(\theta_L)}{\theta_L}\right) > u\left(c(\theta_H)\right) - v\left(\frac{y(\theta_H)}{\theta_L}\right)$$

$$v\left(\frac{y(\theta_H)}{\theta_L}\right) - v\left(\frac{y(\theta_L)}{\theta_L}\right) > u\left(c(\theta_H)\right) - u\left(c(\theta_L)\right)$$

$$v\left(\frac{y(\theta_H)}{\theta_L}\right) - v\left(\frac{y(\theta_L)}{\theta_L}\right) > v\left(\frac{y(\theta_H)}{\theta_H}\right) - v\left(\frac{y(\theta_L)}{\theta_H}\right) \quad \text{(by ICH)}$$

$$\int_{y(\theta_L)}^{y(\theta_H)} v'\left(\frac{y}{\theta_L}\right) dy > \int_{y(\theta_L)}^{y(\theta_H)} v'\left(\frac{y}{\theta_H}\right) dy$$

Therefore, the final line is a sufficient condition for the low-type's incentive compatibility constraint to be slack. The final line in fact holds because of the strict convexity of $v(\cdot)$ and since $\theta_H > \theta_L$.

Lastly, we can decentralize the planner's allocation with a non-linear lump sum tax, $T(y)$:

$$T(y) = \begin{cases} y - c & y \in \{y(\theta_L), y(\theta_H)\} \\ y & \text{otherwise} \end{cases}$$

This satisfies the consumer's dentralized problem:

$$\max_{c,y} u(c) - v\left(\frac{y}{\theta}\right)$$

$$\text{s.t.} \quad c = y - T(y)$$

The optimal allocation can be characterized by:

$$u'(c) = \frac{1}{1 - T'(y)} \cdot \frac{1}{\theta} v'\left(\frac{y}{\theta}\right)$$

where $1/(1 - T')$ reproduces the informational wedge that was present in the planner's constrained optimum. Therefore $T'(y(\theta_H)) = 0$ and $T'(y(\theta_L)) > 0$.

2.2 Static Economy with N≥3 Types

Consider the same environment as above, except now expand the type space to be $\Theta \equiv \{\theta_1, \theta_2, \ldots, \theta_N\}$ for some $N \in [3, \infty) \cap \mathbb{N}$. Let $\theta_1 < \theta_2 < \cdots < \theta_N$ for simplicity. Expanding the type space in this was increases the number of incentive compatibility constraints from two to $N \times (N-1)$. That is, for each type θ_i there are $N-1$ incentive compatibility constraints that must be satisfied in order for the planner's allocation to be implementable as a direct revelation mechanism.

In what follows I will propose a method for reducing the size of the planner's constraint set. I will show that LDICs bind while LUICs are slack. I will then characterize the N-type allocation.

Claim 2.2.1 (Monotonicity)
For each $i > j$, where $\theta_i > \theta_j$, we have $y(\theta_i) > y(\theta_j)$.

Proof. Take some $i, j \in \{1, 2, \ldots, N\}$ such that $i > j$. Now suppose for contradiction that $y(\theta_i) \leq y(\theta_j)$. Consider the two incentive compatibility constraints that ensure agent i does not report he is a $j-$type and vice versa:

$$u(c(\theta_i)) - v\left(\frac{y(\theta_i)}{\theta_i}\right) \geq u(c(\theta_j)) - v\left(\frac{y(\theta_j)}{\theta_i}\right)$$

$$u(c(\theta_j)) - v\left(\frac{y(\theta_j)}{\theta_j}\right) \geq u(c(\theta_i)) - v\left(\frac{y(\theta_i)}{\theta_j}\right)$$

Sum the two constraints to obtain:

$$v\left(\frac{y(\theta_i)}{\theta_j}\right) - v\left(\frac{y(\theta_j)}{\theta_j}\right) \geq v\left(\frac{y(\theta_i)}{\theta_i}\right) - v\left(\frac{y(\theta_j)}{\theta_i}\right)$$

$$\int_{y(\theta_j)}^{y(\theta_i)} v'\left(\frac{y}{\theta_j}\right)\, dy \geq \int_{y(\theta_j)}^{y(\theta_i)} v'\left(\frac{y}{\theta_i}\right)\, dy$$

Since $v(\cdot)$ is *strictly* convex and $\theta_i > \theta_j$, we know:

$$v'\left(\frac{y}{\theta_i}\right) < v'\left(\frac{y}{\theta_j}\right) \qquad \forall\ y > 0$$

But then we arrive at a contradiction:

$$\int_{y(\theta_j)}^{y(\theta_i)} v'\left(\frac{y}{\theta_j}\right)\, dy < \int_{y(\theta_j)}^{y(\theta_i)} v'\left(\frac{y}{\theta_i}\right)\, dy$$

Thus we must have $y(\theta_i) > y(\theta_j)$. That is, production must be monotonically increasing in type in order to satisfy incentive constraints. □

For ease of explication in what follows, define the following two types of constraints.

Definition For any $i \in \{2, 3, \dots, N\}$, a **local downward incentive constraint for the** $i - th$ **agent (LDIC$_i$)** is a incentive compatibility constraint of the form:

$$u(c(\theta_i)) - v\left(\frac{y(\theta_i)}{\theta_i}\right) \geq u(c(\theta_{i-1})) - v\left(\frac{y(\theta_{i-1})}{\theta_i}\right)$$

Definition For any $i \in \{1, 2, \dots, N-1\}$, a **local upward incentive constraint for the** $i - th$ **agent (LUIC$_i$)** is a incentive compatibility constraint of the form:

$$u(c(\theta_i)) - v\left(\frac{y(\theta_i)}{\theta_i}\right) \geq u(c(\theta_{i+1})) - v\left(\frac{y(\theta_{i+1})}{\theta_i}\right)$$

Given monotonicity of production we can now show that local incentive compatibility constraints will be sufficient to characterize the set of $N \times (N-1)$ incentive compatibility constraints. Consider the next four claims, which can be summarized as: (1) if LDIC$_i$ holds then the agent has no incentive misreport *any* lower productivity type, (2) if LUIC$_i$ holds then the agent has no incentive to misreport *any* higher reproductivity, and furthermore (3) LDICs bind while (4) LUICs do not.

Claim 2.2.2 (Local and Global Downward ICs)
If LDIC$_i$ is satisfied for each i and monotonicity is satisfied, then for each i and $j < i$ LDIC$_i$ implies

$$u(c(\theta_i)) - v\left(\frac{y(\theta_i)}{\theta_i}\right) \geq u(c(\theta_j)) - v\left(\frac{y(\theta_j)}{\theta_i}\right)$$

Proof. Suppose that LDIC$_{i+1}$ and LDIC$_i$ both hold. By monotonicity, given by claim 2.2.1, we know that $\theta_{i+1} > \theta_i > \theta_{i-1}$ implies $y_{i+1} > y_i > y_{i-1}$. Therefore since $v(\cdot)$ is strictly increasing:

$$\left.\begin{array}{rl} u(c(\theta_i)) - v\left(\frac{y(\theta_i)}{\theta_i}\right) & \geq u(c(\theta_{i-1})) - v\left(\frac{y(\theta_{i-1})}{\theta_i}\right) \\ \theta_i > \theta_{i-1} \implies & y_i > y_{i-1} \end{array}\right\} \implies u(c(\theta_i)) - v\left(\frac{y(\theta_i)}{\theta_{i+1}}\right) \geq u(c(\theta_{i-1})) - v\left(\frac{y(\theta_{i-1})}{\theta_{i+1}}\right)$$

From LDIC$_{i+1}$ and the above we obtain:

$$\left.\begin{array}{l} u(c(\theta_{i+1})) - v\left(\frac{y(\theta_{i+1})}{\theta_{i+1}}\right) \geq u(c(\theta_i)) - v\left(\frac{y(\theta_i)}{\theta_{i+1}}\right) \\ u(c(\theta_i)) - v\left(\frac{y(\theta_i)}{\theta_{i+1}}\right) \geq u(c(\theta_{i-1})) - v\left(\frac{y(\theta_{i-1})}{\theta_{i+1}}\right) \end{array}\right\} \implies u(c(\theta_{i+1})) - v\left(\frac{y(\theta_{i+1})}{\theta_{i+1}}\right) \geq u(c(\theta_{i-1})) - v\left(\frac{y(\theta_{i-1})}{\theta_{i+1}}\right)$$

If LDIC$_{i-1}$ holds, then we can use the same procedure to show that:

$$u(c(\theta_{i+1})) - v\left(\frac{y(\theta_{i+1})}{\theta_{i+1}}\right) \geq u(c(\theta_{i-2})) - v\left(\frac{y(\theta_{i-2})}{\theta_{i+1}}\right)$$

Therefore,

$$u(c(\theta_i)) - v\left(\frac{y(\theta_i)}{\theta_i}\right) \geq u(c(\theta_j)) - v\left(\frac{y(\theta_j)}{\theta_i}\right)$$

for all $j < i$. □

Claim 2.2.3 (Local and Global Upward ICs)
If LUIC_i is satisfied for each i and monotonicity holds, then for each i and $j > i$.

$$u(c(\theta_i)) - v\left(\frac{y(\theta_i)}{\theta_i}\right) \geq u(c(\theta_j)) - v\left(\frac{y(\theta_j)}{\theta_i}\right)$$

Proof. This proof is almost identical to that above. Suppose that LUIC_i and LUIC_{i+1} both hold. Then LUIC_{i+1} and monotonicity give:

$$u(c(\theta_{i+1})) - v\left(\frac{y(\theta_{i+1})}{\theta_i}\right) \geq u(c(\theta_{i+2})) - v\left(\frac{y(\theta_{i+2})}{\theta_i}\right)$$

The above expression when paired with LUIC_i gives:

$$u(c(\theta_i)) - v\left(\frac{y(\theta_i)}{\theta_i}\right) \geq u(c(\theta_{i+2})) - v\left(\frac{y(\theta_{i+2})}{\theta_i}\right)$$

Which can be iterated to give the result. □

These two claims allow us to restrict the set of incentive compatibility constraints to local ICs. The next two lemmas estabilish that we may restrict the set to local downward incentive compatibility constraints.

Claim 2.2.4
If monotonicity is satisfied, LDICs are binding for each type i.

Proof. Proceed by contradiction. Suppose the LDIC is slack for some type i:

$$u(c(\theta_i)) - v\left(\frac{y(\theta_i)}{\theta_i}\right) > u(c(\theta_{i-1})) - v\left(\frac{y(\theta_{i-1})}{\theta_i}\right)$$

But then the planner can increase $y(\theta_i)$ by some $\varepsilon_i > 0$ until LDIC_i binds. Denote $\widehat{y}(\theta_i) = y(\theta_i) + \varepsilon_i$ and write the new LDIC_i as:

$$u(c(\theta_i)) - v\left(\frac{\widehat{y}(\theta_i)}{\theta_i}\right) = u(c(\theta_{i-1})) - v\left(\frac{y(\theta_{i-1})}{\theta_i}\right)$$

However, increasing $y(\theta_i)$ may cause LDIC_{i+1} to become slack, in which case we must choose some $\varepsilon_{i+1} > 0$ in order to construct $\widehat{y}(\theta_{i+1}) = y(\theta_{i+1}) + \varepsilon_{i+1}$ that ensures LDIC_{i+1} binds:

$$u(c(\theta_{i+1})) - v\left(\frac{y(\theta_{i+1})}{\theta_{i+1}}\right) > u(c(\theta_{i+1})) - v\left(\frac{\widehat{y}(\theta_{i+1})}{\theta_{i+1}}\right) = u(c(\theta_i)) - v\left(\frac{\widehat{y}(\theta_i)}{\theta_i}\right)$$

And thus LDIC_{i+1} binds.

We can iteratively continue to increase each type's production until we construct the $\{\widehat{y}_j\}_{j=2}^N$ that ensures LDIC_j holds for each $j \in \{i, i+1, \ldots, N\}$. Lastly this new production contract is resource feasible since we have not changed consumption contracts. □

Claim 2.2.5
If monotonicity is satisfied, LUICs are slack for each type i.

Proof. Proceed by contradiction. Suppose the LUIC is binding for some type i:

$$u(c(\theta_i)) - v\left(\frac{y(\theta_i)}{\theta_i}\right) = u(c(\theta_{i+1})) - v\left(\frac{y(\theta_{i+1})}{\theta_i}\right)$$

By the above claim 2.2.4 we know that monotonicity is sufficient to ensure LDIC$_{i+1}$ binds.

$$u(c(\theta_{i+1})) - v\left(\frac{y(\theta_{i+1})}{\theta_{i+1}}\right) = u(c(\theta_i)) - v\left(\frac{y(\theta_i)}{\theta_{i+1}}\right)$$

Summing the binding LDIC$_{i+1}$ and binding LUIC$_i$ we obtain:

$$v\left(\frac{y(\theta_{i+1})}{\theta_{i+1}}\right) - v\left(\frac{y(\theta_i)}{\theta_{i+1}}\right) = v\left(\frac{y(\theta_{i+1})}{\theta_i}\right) - v\left(\frac{y(\theta_i)}{\theta_i}\right)$$

$$\int_{y(\theta_i)}^{y(\theta_{i+1})} v'\left(\frac{y}{\theta_{i+1}}\right)\, dy = \int_{y(\theta_i)}^{y(\theta_{i+1})} v'\left(\frac{y}{\theta_i}\right)\, dy$$

But the last line contradicts the following implication of strict convexity and $\theta_{i+1} > \theta_i$:

$$v'\left(\frac{y}{\theta_{i+1}}\right) < v'\left(\frac{y}{\theta_i}\right) \qquad \forall\, y > 0$$

Hence LUIC$_i$ is slack for each $i \in \{1, \ldots, N-1\}$. □

Before considering a social planner's problem, I will provide one more result. It is a characterization of incentive compatible consumption schedules, but does not bear incidence upon the constraint set as the last set of results have. It is provided for completeness.

Claim 2.2.6
Consumption is monotonically increasing with type: if $\theta_i > \theta_j$ then $c(\theta_i) > c(\theta_j)$ for all $i > j$.

Proof. Rewrite the incentive compatibility constraint of an agent type j who reports that he is an agent type i, where $i > j$:

$$u(c(\theta_j)) - u(c(\theta_i)) \geq v\left(\frac{y(\theta_j)}{\theta_j}\right) - v\left(\frac{y(\theta_i)}{\theta_j}\right)$$

Since we know that production is monotonically increasing with type, $y(\theta_j) > y(\theta_i)$, and that $v''(\cdot) > 0$, then we know the RHS is strictly positive. Therefore $u(c(\theta_j)) > u(c(\theta_i))$ which implies $c(\theta_j) > c(\theta_i)$. □

Now that we have established the last series of claims we can define and solve a "relaxed" program:

$$\max_{c(\theta),y(\theta)} \sum_i \pi(\theta_i) \left[u\Big(c(\theta_i)\Big) - v\Big(\frac{y(\theta_i)}{\theta_i}\Big) \right]$$

s.t.

$$\forall\ i > 1 \quad u\Big(c(\theta_i)\Big) - v\Big(\frac{y(\theta_i)}{\theta_i}\Big) \geq u\Big(c(\theta_{i-1})\Big) - v\Big(\frac{y(\theta_{i-1})}{\theta_i}\Big) \qquad (\text{LDIC}_i, \pi(\theta_i)\ \mu(\theta_i))$$

$$\sum_i \pi(\theta_i) c(\theta_i) \leq \sum_i \pi(\theta_i) y(\theta_i) \qquad (\text{RC}, \lambda)$$

The first order conditions for this relaxed problem are:

$$(1 + \mu(\theta_N))\ u'(c(\theta_N)) = \lambda$$

$$\frac{1}{\theta_N} v'\left(\frac{y(\theta_N)}{\theta_N}\right) = \lambda$$

$$\left(1 + \mu(\theta_i) - \mu(\theta_{i+1}) \frac{\pi(\theta_{i+1})}{\pi(\theta_i)}\right)\ u'(c(\theta_i)) = \lambda$$

$$(1 - \mu(\theta_i)) \frac{1}{\theta_i} v'\left(\frac{y(\theta_i)}{\theta_i}\right) - \mu(\theta_{i+1}) \frac{\pi(\theta_{i+1})}{\pi(\theta_i)} \frac{1}{\theta_{i+1}} v'\left(\frac{y(\theta_i)}{\theta_{i+1}}\right) = \lambda$$

We have already shown that $y(\theta_{i+1}) > y(\theta_i)$ for all $i < N$, and that LUIC_I is slack for all $i < N$. We can now characterize how the constrained optimum distorts the optimal allocations of agents away from the full information optimum.

P3': No Distortion At The Top

As in the two-type case, the N-type optimal mechanism also does not distort the allocations of the most productive agents. To see this combine the FOCs of consumption and output for any type-N agent and obtain the full information consumption-labor condition:

$$u'(c(\theta_N)) = \frac{1}{\theta_N} v'\left(\frac{y(\theta_N)}{\theta_N}\right)$$

P4': Distortion At The Bottom

Again, this environment exhibits that all other agents ($i < N$) have distorted allocations compared to the full information optimum. To see this combine the FOCs of consumption and output for any type $i < N$ agent and obtain:

$$\underbrace{\left[\frac{1 + \mu(\theta_i) - \mu(\theta_{i+1}) \frac{\pi(\theta_{i+1})}{\pi(\theta_i)}}{1 + \mu(\theta_i) - \mu(\theta_{i+1}) \frac{\pi(\theta_{i+1})}{\pi(\theta_i)} \frac{\theta_i}{\theta_{i+1}} \frac{v'\left(\frac{y(\theta_i)}{\theta_{i+1}}\right)}{v'\left(\frac{y(\theta_i)}{\theta_i}\right)}}\right]}_{\xi_i} u'(c(\theta_i)) = \frac{1}{\theta_i} v'\left(\frac{y(\theta_i)}{\theta_i}\right)$$

where ξ_i is the distortion introduced by the planner. Therefore this optimality condition exhibits distortion away from the full information optimum.

As we did before, we can decentralize the constrained efficient allocation by using a non-linear lump sum tax on consumers. The consumers' decentralized problem is:

$$\max_{c,y}\ u(c) - v\left(\frac{y}{\theta}\right)$$

$$\text{s.t.} \quad c = y - T(y)$$

The optimal allocation can be characterized by:

$$u'(c) = \frac{1}{1 - T'(y)} \cdot \frac{1}{\theta} v'\left(\frac{y}{\theta}\right)$$

where $1/(1 - T')$ reproduces the informational wedge that was present in the planner's constrained optimum. Therefore a necessary condition for lump sum taxation to implement the constrained efficient allocation is for $T'(y(\theta_N)) = 0$ and $T'(y(\theta_i)) = \xi_i$ for all $i < N$.

2.3 Static Economy with a Continuum of Types

For now, see Fudenberg and Tirole's text book, or Ilya Segal's lecture notes for a good exposition of the methods used in the analysis of a social planner's problem with a continuum of types.

2.4 Dynamic Economy with Finite Types

In this section I will discuss optimality in a simplified version of the environment analyzed by Golosov, Kocherlakota, and Tsyvinski (RES 2003). In particular I will focus on the environment of Golosov and Tsyvinksi (RES 2004), which I describe as follows. I will maintain the environment of the previous section with $N = 2$ types. However, now we will consider a finite time problem with two periods, denoted $t = 0, 1$. In the first period agents receive an endowment, e. In the second period agents receive an iid stochastic productivity shock $\theta \in \Theta = \{0, \overline{\theta}\}$. I will draw the distinction between types by calling $\theta = 0$ types *disabled* and $\theta = \overline{\theta}$ types *abled*. That is, the low shock excludes the agent from production: $y = 0 \cdot \ell = 0$. Therefore first period utility is simply $u(c_0)$. Second period utility across types will be

$$U(c_1(\theta), \ell_1(\theta)) = \begin{cases} u(c_1(\theta)) - v(\ell_1(\theta)) & \text{for } \theta = \overline{\theta} \\ u(c_1(\theta)) & \text{for } \theta = 0 \end{cases}$$

The shock is private information to the agent, as before, and the probability of being the low type is denoted p. Lastly, agents discount the future with discount factor β, which for some gross interest rate $R > 0$ we will assume satisfies $\beta R = 1$.

Notice that this setup simplifies incentive compatibility constraints in an important way. Low types are incapable of reporting that they are high types, therefore eliminating the LUIC altogether.

For notational ease, let

$$c_1(\theta) \equiv \begin{cases} c_d & \text{if } \theta = 0 \\ c_a & \text{if } \theta = \overline{\theta} \end{cases} \qquad \text{and} \qquad y(\theta) \equiv \begin{cases} 0 & \text{if } \theta = 0 \\ y_a & \text{if } \theta = \overline{\theta} \end{cases}$$

Let's first solve a social planner's problem under full information.

$$\max_{c_a, c_d, y_a} u(c_0) + \beta \left(p u(c_d) + (1 - p) \left[u(c_a) - v\left(\frac{y_a}{\overline{\theta}}\right) \right] \right)$$

$$\text{s.t.} \quad c_0 + R^{-1}\Big(pc_d + (1-p)c_a\Big) \leq e + R^{-1}(1-p)y_a \qquad (RC, \lambda)$$

The first order conditions for the full information problem are:

$$u'(c_0) = \lambda$$

$$\beta p u'(c_d) = \beta p \lambda$$

$$\beta(1-p)u'(c_a) = \beta(1-p)\lambda$$

$$\beta(1-p)\frac{1}{\overline{\theta}}v'\left(\frac{y_a}{\overline{\theta}}\right) = \beta(1-p)\lambda$$

These FOCs suggest the following characterization of the full information optimum:

$$u'(c_0) = \beta R u'(c_d) = \beta R u'(c_a) = \beta R \frac{1}{\overline{\theta}} v'\left(\frac{y_a}{\overline{\theta}}\right)$$

The characterization shows that consumption is equalized across shocks. Consumption across time satisfies a standard Euler equation. The characterization furthermore shows that there is no distortion at the top, and low types are better off than in autarky since they would have not been able to consume otherwise.

Now consider the environment with private information. We must add an incentive constraint to make sure that high types do not report that they are low types and attempt to receive insurance despite their ability to produce. Consider the following social planner's problem:

$$\max_{c,y} u(c_0) + \beta\left(pu(c_d) + (1-p)\left[u(c_a) - v\left(\frac{y_a}{\overline{\theta}}\right)\right]\right)$$

$$\text{s.t.} \quad c_0 + R^{-1}\Big(pc_d + (1-p)c_a\Big) \leq e + R^{-1}(1-p)y_a \qquad (RC, \lambda)$$

$$u(c_a) - v\left(\tfrac{y_a}{\overline{\theta}}\right) \geq u(c_d) \qquad (IC, \beta(1-p)\mu)$$

The first order conditions for the private information problem are:

$$u'(c_0) = \lambda$$

$$\left(1 - \mu\frac{1-p}{p}\right)u'(c_d) = \lambda$$

$$(1+\mu)u'(c_a) = \lambda$$

$$(1+\mu)\frac{1}{\overline{\theta}}v'\left(\frac{y_a}{\overline{\theta}}\right) = \lambda$$

By combining the last two FOCs, we immediately see that there is no distortion at the top:

$$u'(c_a) = \frac{1}{\overline{\theta}}v'\left(\frac{y_a}{\overline{\theta}}\right)$$

In this environment, assessing whether there is distortion for low types is equivalent to characterizing intertemporal distortions since under the full information optimum $u'(c_0) = u'(c_d)$. In order to characterize potential distortions to the intertemporal margin, we will derive the following **inverse Euler equation**.

Claim 2.4.1 (Inverse Euler Equation)
A constrained efficient allocation satisfies an Inverse Euler Equation:

$$\frac{1}{u'(c_0)} = \beta R \left(\frac{p}{u'(c_d)} + \frac{1-p}{u'(c_a)} \right)$$

Which characterizes intertemporal distortions relative to the Standard Euler Equation as:

$$u'(c_0) \leq \beta R \Big(pu'(c_d) + (1-p)u'(c_a) \Big)$$

with equality if $u'(c_d) = u'(c_a)$ as in the full information optimum.

Proof. Multiply the FOC with respect to the high type's consumption by $(1-p)$ and rewrite as:

$$(1-p)(1+\mu) = (1-p)\frac{u'(c_0)}{u'(c_a)}$$

Multiply the FOC with respect to the low type's consumption by p and rewrite as:

$$p\left(1 - \mu\frac{1-p}{p}\right) = p\frac{u'(c_0)}{u'(c_d)}$$

By summing the two FOCs we can rearrange to obtain the inverse Euler equation. Lastly, since θ is a random variable, then so is $u'(c(\theta))$. We may therefore apply Jensen's Inequality on the convex function $1/u(c(\theta))$:

$$\frac{1}{u'(c_0)} = \mathbb{E}_\theta \left[\frac{1}{u'(c(\theta))} \right] \geq \frac{1}{\mathbb{E}_\theta \left[u'(c(\theta)) \right]}$$

Therefore we obtain the intemporal distortion as desired, noting that $\beta R = 1$. □

The claim tells us that the constrained efficient allocation distorts the implicit savings margin. Due to informational frictions, an agent with higher savings can more profitably report he is the low type if he receives the high shock. Therefore the planner discourages high types from misreporting by making this deviation less profitable. This is optimal because the planner can create more insurance by inducing high types to not misreport and work more.

Lastly, notice that $c_a > c_d$ by the same argument in the previous section.

2.4.1 Implementation

In this section I will consider two implementation strategies. I will first show the lump sum taxation implementation of the previous sections does not yield the constrained efficient allocation. I will then follow two papers, Golosov and Tsyvinksi (RES 2004) and Kocherlakota (EMA 2003), to provide two implementations that do yield the constrained efficient allocation.

To fix notation, let c_a be an able agent's consumption decision, c_d be the disabled agent's consumption decision, y be the able agent's output decision.

The basic decentralized agent's maximization problem is:

$$\max_{c_a,c_d,y,s} u(c_0)+\beta\left(pu(c_d)+(1-p)\left[u(c_a)-v\left(\frac{y}{\overline{\overline{\theta}}}\right)\right]\right)$$

$$\begin{aligned} \text{s.t.}\quad c_0+s &\leq e && (BC_0,\lambda_0)\\ c_a &\leq y+Rs && (BC_1,\lambda_a)\\ c_d &\leq Rs && (BC_1,\lambda_d) \end{aligned}$$

where s is the savings of the agent. First order conditions give us the following characterization of the decentralized allocation, before considering tax policy.

$$u'(c_a)=\frac{1}{\overline{\overline{\theta}}}v'\left(\frac{y}{\overline{\overline{\theta}}}\right)$$

$$u'(c_0)=\beta R\Big[pu'(c_d)+(1-p)u'(c_a)\Big]$$

The characterization implies there is no distortion for high types and there is no intertemporal distortion. But then the inverse Euler equation is not satisfied. Therefore tax policy must distort the savings decision.

2.4.2 Implementation: Linear Taxation

Consider a linear savings tax, τ, and output contingent lump sum taxes, $\{T_a, T_d\}$. The contingent lump sum taxes allow the government to try taxing able and disabled agents at different amounts. We can then rewrite the agent's maximization problem as:

$$\max_{c_a,c_d,y,s} u(c_0)+\beta\left(pu(c_d)+(1-p)\left[u(c_a)-v\left(\frac{y}{\overline{\overline{\theta}}}\right)\right]\right)$$

$$\begin{aligned} \text{s.t.}\quad c_0+s &\leq e\\ c_a &\leq y+(1-\tau)Rs+T_a\mathbb{I}[y>0]+T_d\mathbb{I}[y=0]\\ c_d &\leq (1-\tau)Rs+T_d \end{aligned}$$

The first order conditions still show that there is no distortion for high types. The Euler equation in this environment is:

$$u'(c_0)=\beta R(1-\tau)\Big[pu'(c_d)+(1-p)u'(c_a)\Big]$$

We will not attempt to construct the policy as to implement the constrained efficient allocation. First denote the constrained efficient allocation with a $*$ superscript, and then define policies and savings as:

$$s^*=e-c_0^*$$

$$T_a = y^* + (1-\tau)s^* - c_a^*$$

$$T_d = (1-\tau)s^* - c_d^*$$

$$\tau = 1 - \frac{u'(c_0^*)}{pu'(c_d^*) + (1-p)u'(c_a^*)}$$

Whereas these first order conditions solve the program, they are not sufficient. To see this, let's show that a deviation strategy is profitable. In particular, consider the maximization problem of an agent who decides at $t = 0$ that he will claim he is the low type at $t = 1$ regardless of his true realization of ability shock. We can write the deviator's maximization problem as:

$$\max_{c_d, y, s} u(c_0) + \beta u(c_d)$$

$$\text{s.t.} \quad c_0 + s \leq e$$

$$c_d \leq (1-\tau)Rs + T_d$$

The first order conditions for this problem suggest the following characterization of the "deviation optimum":

$$u'(c_0) = \beta R(1-\tau)u'(c_d)$$

Recall that since high types consume more in the constrained efficient allocation, $c_a^* > c_d^*$, then the concavity of the utility function implies $u'(c_a^*) < u'(c_d^*)$. This property enables us to show that the characterization of the optimum under deviation is not satisfied when evaluated at the constrained efficient allocation:

$$u'(c_0^*) = \beta R(1-\tau)\Big[pu'(c_d^*) + (1-p)u'(c_a^*)\Big] < \beta R(1-\tau)u'(c_d^*)$$

Therefore, when an agent surely plays the deviation strategy the constrained efficient allocation will not satisfy the agent's first order conditions. When the agent deviates in this decentralized environment, he will optimally choose a different allocation than the constrained efficient one. Under the decentralized allocation, the agent will consume less and save more in the first period relative to the constrained efficient allocation ($c_0 < c_0^*, s > s^* = e - c_0^*$), and will therefore consume more in the second period ($c_d > c_d^*$).

Lastly, since the incentive compatibility constraint was binding under the constrained efficient allocation, the deviation strategy induces higher second period consumption c_d that will violate the incentive compatibility constraint:

$$u(c_a^*) + v\left(\frac{y_a^*}{\overline{\theta}}\right) = u(c_d^*) < u(c_d)$$

Because we can show that deviations are profitable, we have shown that the linear tax cannot implement the constrained efficient allocation.

The result that a deviation strategy precludes linear income taxation from decentralizing the constrained efficient allocation is due to linear taxation not sufficiently distorting the savings decision. The inverse Euler equation shows that the social planner wishes to discourage savings and thereby decrease the profitability of high agents choosing to misreport. In the decentralization with linear income taxes we are able to explicitly see why: any agent could profitably deviate from truthtelling by saving a high amount in the first period (relative to the constrained efficient level) and then collecting the low type's transfer while not working.

2.4.3 Implementation: Capital Taxation

Kocherlakota (EMA 2003) considers a policy in this decentralized environment that implements the constrained efficient allocation: savings taxation. The idea of this policy is to distort the savings decision by directly taxing savings and lump sum transfering the tax revenue to disabled agents. Furthermore, by conditioning the savings tax on output we can punish agents who claim disability but accumulate high savings. The failure of linear taxation was that it did not have to discriminatory power to prevent agents from saving high amounts and collecting disability insurance instead of suffering the disutility of labor.

Kocherlakota denotes savings taxes conditional on zero output and positive output as $\{\tau_{y=0}, \tau_{y>0}\}$, repsectively. The conditional lump sum taxes are denoted by $\{T_{y=0}, T_{y>0}\}$. The agent's ex ante maximization problem is:

$$\max_{c_a, c_d, y, s} u(c_0) + \beta\Big(pu(c_d) + (1-p)\Big[u(c_a) - v\left(\frac{y}{\overline{\theta}}\right)\Big]\Big)$$

$$\begin{aligned} \text{s.t.} \quad c_0 + s &\leq e \\ c_a &\leq \Big[(1-\tau_{y>0})Rs + y + T_{y>0}\Big]\mathbb{I}(y>0) + \Big[(1-\tau_{y=0})Rs + T_{y=0}\Big]\mathbb{I}(y=0) \\ c_d &\leq (1-\tau_{y=0})Rs + T_{y=0} \end{aligned}$$

In order to construct the tax schedule that implements the constrained efficient allocation, consider the following two subproblems. First, consider the maximization problem of an agent who will surely be disabled:

$$V_{y=0} = \max_{c_d, s} u(c_0) + \beta u(c_d)$$

$$\begin{aligned} \text{s.t.} \quad c_0 + s &\leq e \\ c_d &\leq (1-\tau_{y=0})Rs + T_{y=0} \end{aligned}$$

The Euler equation for this subproblem is:

$$u'(c_0) = \beta R(1-\tau_{y=0})u'(c_d)$$

Second, consider the maximization problem of an agent who will surely be able and produces:

$$V_{y>0} = \max_{c_a, y, s} u(c_0) + \beta\Big[u(c_a) - v\left(\frac{y}{\overline{\theta}}\right)\Big]$$

$$\begin{aligned} \text{s.t.} \quad c_0 + s &\leq e \\ c_a &\leq (1-\tau_{y>0})Rs + y + T_{y>0} \end{aligned}$$

The Euler equation for this second subproblem is then:

$$u'(c_0) = \beta R(1 - \tau_{y>0})u'(c_a)$$

In order to ensure that the savings taxes are consistent with the constrained efficient consumption quantities *and* individual optimization (when agents produce if they can), set the savings taxes as follows:

$$\tau_{y>0} = 1 - \frac{u'(c_0^*)}{\beta R u'(c_a^*)} \qquad \text{and} \qquad \tau_{y=0} = 1 - \frac{u'(c_0^*)}{\beta R u'(c_d^*)}$$

where the superscript $*$ denotes the constrained efficient allocation quantities as before. Notice that Kocherlakota constructs the savings taxes by rearranging the Euler equations from the two subproblems.

Furthermore, the Euler equations from the two subproblems can be multiplied by their respective probability of ability shock and summed in order to obtain the Euler equation of the original ex ante maximization problem.

Next define the lump sum transfers as follows:

$$\begin{aligned} T_{y=0} &= c_d^* - (1 - \tau_{y=0})Rs^* \\ T_{y>0} &= c_a^* - (1 - \tau_{y>0})Rs^* - y^* \\ s^* &= e - c_0^* \end{aligned}$$

By construction of this tax policy the constrained efficient allocation solves the ex ante maximization problem: both the Euler equation and inverse Euler equation are satisfied. However we must show that the FOC characterizes the optimum. As in the case of linear taxation, we must show that the tax policy rules out the profitability of a "deviation" in which an agent always reporting disability regardless of true type. To show this notice that under the constrained efficient allocation the incentive compatibility constraint holds with equality. But then we have that:

$$V_{y>0} = u(c_a^*) - v\left(\frac{y^*}{\overline{\theta}}\right) \;=\; u(c_d^*) = V_{y=0}$$

Therefore the incentive compatibility constraint holds with equality, able agents are indifferent between deviating and producing, and therefore deviation is not optimal.

Notice two properties of this tax policy. The first property is that the expected savings tax is zero. To see this write the inverse Euler equation:

$$\begin{aligned} \frac{1}{u'(c_0^*)} &= p\frac{1}{u'(c_d^*)} + (1-p)\frac{1}{u'(c_a^*)} \\ 1 &= p(1 - \tau_{y=0}) + (1-p)(1 - \tau_{y>0}) \\ 1 &= p + (1-p) - \Big(p\tau_{y=0} + (1-p)\tau_{y>0}\Big) \\ 0 &= p\tau_{y=0} + (1-p)\tau_{y>0} \end{aligned}$$

Furthermore since $c_a^* > c_d^*$ and $u'(c_a^*) < u'(c_d^*)$, we know that $\tau_{y=0} > \tau_{y>0}$. But since taxes are zero in expectations, this last condition requires that $\tau_{y=0} > 0 > \tau_{y>0}$. In words, in order to implement the constrained efficient allocation, the savings tax $\tau_{y=0}$ will serve as a punishment to those who potentially untruthfully claim disability (e.g. $y = 0$) and the savings tax $\tau_{y>0}$ begin negative serves as an incentive to able types to produce.

The second property is that the expected lump sum transfer is independent of the savings tax. To see this, take expectations:

$$
\begin{aligned}
\mathbb{E}_y[T_y] &= pT_{y=0} + (1-p)T_{y>0} \\
&= p\Big\{c_d^* - (1-\tau_{y=0})Rs^*\Big\} + (1-p)\Big\{c_a^* - (1-\tau_{y>0})Rs^* - y^*\Big\} \\
&= p\Big\{c_d^* - (1-\tau_{y=0})R(e-c_0^*)\Big\} + (1-p)\Big\{c_a^* - (1-\tau_{y>0})R(e-c_0^*) - y^*\Big\} \\
&= \Big\{pc_d^* + (1-p)(c_a^* - y^*)\Big\} - R(e-c_0^*)\Big\{p(1-\tau_{y=0}) + (1-p)(1-\tau_{y>0})\Big\} \\
&= pc_d^* + (1-p)(c_a^* - y^*)
\end{aligned}
$$

Therefore the expected transfer is equal to expected net consumption.

2.4.4 Implementation: Asset Testing

Golosov and Tsyvinksi (RES 2004) consider another policy in this decentralized environment that implements the constrained efficient allocation: asset testing. The essence of this policy is to condition the lump sum tax not only on observed production but also on savings. If an agent saves more than a specified level then they should be taxed in a lump sum manner. If they save less than or equal to the constrained efficient level, the agent should be given a transfer. This eliminates deviations induced by high savings.

The authors define an *asset-tested disability insurance program* as a set of lump sum transfers that are a function of output and savings $\{T_a(y,s), T_d(y,s)\}$, and an asset test, $\{\overline{s}\}$, that specifies a cut-off rule for savings when determining whether an agent should receive a tax or transfer. The agent's ex ante maximization problem is as follows:

$$
\max_{c_a, c_d, y, s} u(c_0) + \beta\Big(pu(c_d) + (1-p)\Big[u(c_a) - v\left(\frac{y}{\overline{\theta}}\right)\Big]\Big)
$$

$$
\begin{aligned}
\text{s.t.} \quad c_0 + s &\le e \\
c_a &\le Rs + y + T_a(y,s) + T_d(y,s) \\
c_d &\le Rs + T_a(y,s) + T_d(y,s)
\end{aligned}
$$

where I have further defined:

$$
T_a(y,s) \equiv T_a \mathbb{I}(y > 0 \vee s > \overline{s})
$$

$$
T_d(y,s) \equiv T_d \mathbb{I}(y = 0 \wedge s \le \overline{s})
$$

Claim 2.4.2
For any constrained efficient allocation $\{c_0^*, c_d^*, c_a^*, y^*\}$ there exists a policy $\{\overline{s}, T_a(y,s), T_d(y,s)\}$ such that the competitive equilibrium is the constrained efficient allocation.

Proof. Consider a policy $\{\overline{s}, T_a(y,s), T_d(y,s)\}$ in which the asset-test is defined by:

$$\overline{s} \equiv e - c_0^*$$

And the transfers solve budget constraints at the constrained efficient allocations:

$$T_a = c_a^* - R\overline{s} - y^*$$

$$T_d = c_d^* - R\overline{s}$$

We must show that under this policy, high types will weakly prefer choosing output and savings such that $y > 0$ and $s > \overline{s}$ than otherwise. Put in other words, the high types will weakly prefer to work and save "large" sums as to be taxed T_a, rather than to not work and save "small" sums and collect the transfer T_d. Since there are two ways agents could collect the transfer under the decentralized problem without policy intervention, e.g. a double deviation through output and savings decisions, the lump-sum taxes control for both sources of deviation. However, in order to check that the proposed policy discourages deviations we must check utilities of deviation for each combination of output and savings decisions: $\{(y > 0, s > \overline{s}), (y = 0, s > \overline{s}), (y > 0, s \leq \overline{s}), (y = 0, s \leq \overline{s})\}$.

<u>Case 1:</u> $(y = 0, s \leq \overline{s})$
Suppose the agent decides to not produce $(y = 0)$ and to save at most $\overline{s}$ regardless of his type at $t = 1$. In effect this agent decides to receive disability benefits. Then the agent solves:

$$\max_{c_d, s} u(c_0) + \beta u(c_d)$$

$$\text{s.t.} \quad c_0 + s \leq e$$

$$c_d \leq Rs + T_d$$

$$s \leq \overline{s}$$

Notice that if the agent saves an amount $\overline{s}$ then the constrained efficient allocation solves his maximization problem. Then the characterization of the optimum is:

$$u'(c_0^*) < \beta R u'(c_d^*)$$

The characterization tells us that the agent would prefer to save more, but the planner distorts the savings decision downward.

Now suppose the agent saves less than $\overline{s}$, call it $\tilde{s} = \overline{s} - \varepsilon$ for some $e > 0$. Then the agent consumes more in the first period $\tilde{c}_0 = c_0^* + \varepsilon > c_0^*$, and less in the second period $\tilde{c}_d = c_d^* - R\varepsilon < c_d^*$. But then we know that saving less reduces utility since,

$$u'(\tilde{c}_0) < u'(c_0^*) < \beta R u'(c_d^*) < \beta R u'(\tilde{c}_d)$$

Therefore the agent will save $\overline{s}$ when he produces zero output and is restricted to savings of $s \leq \overline{s}$.

Case 2: $(y = 0, s > \overline{s})$
Take some $\varepsilon > 0$ and suppose the agent saves $\tilde{s} = \overline{s} + \varepsilon$. Denote consumption under this savings strategy as $(\tilde{c}_0, \tilde{c}_d)$. We know that $\tilde{c}_0 < c_d^*$ since the agent saves more relative to the constrained efficient level. Furthermore, $\tilde{c}_d = c_d^* + R\varepsilon + T_a$ by construction of taxes. We want to show that $\tilde{c}_d < c_d^*$. To do so, notice that:

$$\begin{aligned} c_a^* &< y^* + R(\overline{s} - \varepsilon) \\ R\varepsilon + c_a^* &< y^* + R\overline{s} \\ R\varepsilon + T_a &< 0 \end{aligned}$$

It follows from negativity of $R\varepsilon + T_a$ that $\tilde{c}_d < c_d^*$. Therefore by saving an epsilon more, the agent lowers his consumption in both periods. This is clearly suboptimal given the tax structure. The agent will prefer to save $\overline{s}$.

Cases 3 & 4: $(y > 0, s \leq \overline{s})\&(y > 0, s > \overline{s})$
Suppose the agent decides to produce $(y > 0)$ given he is an able type. Then the agent solves:

$$\max_{c_a, s} \; u(c_0) + \beta\Big[u(c_a) - v\Big(\frac{y}{\overline{\theta}}\Big)\Big]$$

$$\begin{aligned} \text{s.t.} \quad c_0 + s &\leq e \\ c_a &\leq Rs + T_a \end{aligned}$$

When savings are not restricted, the intertemporal and intratemporal conditions for this maximization problem are:

$$u'(c_0) = \beta R u'(c_a) \qquad \text{and} \qquad u'(c_a) = \frac{1}{\overline{\theta}} v'\left(\frac{y}{\overline{\theta}}\right)$$

Given the construction of the taxes, the intratemporal condition is not distorted and implements the constrained efficient allocation for c_a^* and y^*. This is turn implies that the savings and first period consumption levels will also be constrained efficient.

Since we found the cases in which the policy implements the constrained efficient allocation, we know the incentive compatibility constraint is satisfied with equality. Therefore agents are indifferent between producing and not producing when they are able. □

2.5 Exercises: Prelim Questions

To be completed.

3 Dynamic Contracting with Taste Shocks

In this section I will consider dynamic environments in which marginal utility is stochastic and privately observed. We will focus on efficient risk pooling mechanisms in which a social planner will provide incentives consistent with cross-subsidizing risk. I will first consider Diamond and Dybvig's (JPE 1983) model of liquidity risk and cast the types of issues that arise in this literature. I will then set up the recursive dynamic program in Atkeson and Lucas (RES 1992) and provide their immiseration result. Lastly I will consider Farhi and Werning's (JPE 2007) reformulation of the Atkeson and Lucas environment and provide results on how "mean-reverting" mechanisms eliminate immiseration.

3.1 Diamond and Dybvig (JPE 1983)

Diamond and Dybvig (JPE 1983) consider the problem of liquidity provision when agents' wealth is invested in a long term and illiquid asset.

There is a continuum on $[0, 1]$ of ex ante identical agents. There are three periods in which agents act, denoted $t = 0, 1, 2$. In the first period, $t = 0$, agents invest/deposit their endowment of one unit of consumption. At $t = 1$ agents receive a preference shock $\theta \in \{\theta_I, \theta_P\}$, where I denotes an "impatient" type and P denotes a "patient" type. The preference shock induces preferences over the timing of consumption: a patient agent values consumption at both $t = 1, 2$, while an impatient agent only values consumption at $t = 1$. The preference shock θ_I can be thought of as an exogenous need for liquidity. Accordingly, at $t = 1$ an agent reports whether he would prefer to withdraw his deposit at zero net interest earned or to allow the deposit to sit until $t = 2$ and earn gross interest $R > 1$.

Let $\theta_I = 1$ and $\theta_P = 0$. Denote the probability of an agent receiving shock θ_I as λ and the probability of receiving shock θ_P as $(1 - \lambda)$. Agents' payoffs are then:

$$u\Big(c_1, c_2; \theta\Big) = \theta u(c_1) + (1 - \theta)\rho u(c_1 + c_2)$$

where $u(\cdot)$ is twice continuously differentiable, increasing, strictly concave, satisfies Inada conditions, and has a coefficient of relative risk aversion $-cu''(c)/u'(c) > 1$. Furthermore assume that $R^{-1} < \rho \leq 1$.

The liquidity shocks are private information. Accordingly agents report their type to an ex ante utilitarian social planner who then allocates resources based on reports. Restricting attention to direct revelation mechanisms, impose the following incentive constraints that ensure agents do not misreport their type:

$$u(c_1(\theta_I)) \geq u(c_1(\theta_P))$$

$$\rho u(c_1(\theta_P) + c_2(\theta_P)) \geq \rho u(c_1(\theta_I))$$

These ICs imply that:

$$\rho u(c_1(\theta_P) + c_2(\theta_P)) \geq \rho u(c_1(\theta_P))$$

Before considering the optimal allocation of resources as decided by a social planner, let's study the agent's decisions under autarky. Suppose as Diamond and Dybvig do, that the autarkic agent may invest in the long term technology at $t = 0$ and may liquidate it at $t = 1$. If the agent decides to liquidate, then he receives zero net interest. Since $R > 1$, the agent will invest his entire endowment in the long term technology. Therefore if he discovers he is the patient type at $t = 1$ then he will wait to consume $c_2 = R$ at $t = 2$. If he discovers he is the impatient type, then the agent will liquidate the entire investment since he does not value last period consumption. Hence impatient agents consume $c_1 = 1$. Now let's describe how a social planner can better insure agents.

An ex ante utilitarian social planner would allocate consumption according to the following maximization problem. In this problem I will exclude the incentive constrains and show that the allocation that solves the program already satisfies the ICs.

$$\max_{c_1(\theta_I),c_1(\theta_P),c_2(\theta_P)} \quad \lambda u(c_1(\theta_I)) + (1-\lambda)\rho u(c_1(\theta_P) + c_2(\theta_P))$$

$$\text{s.t.} \quad \lambda c_1(\theta_I) + (1-\lambda)\Big[c_1(\theta_P) + R^{-1}c_2(\theta_P)\Big] \quad \leq 1$$

First order conditions are necessary and sufficient to characterize the optimum. The FOC shows that the planner optimally equates the marginal rate of substitution between types to the discounted gross interest on the long maturity technology.

$$\frac{u'(c_1(\theta_I))}{u'(c_2(\theta_P))} = \rho R$$

Now I will more fully characterize the optimal allocation.

Claim 3.1.1
The optimal allocation satisfies $0 = c_1(\theta_P) < 1 < c_1(\theta_I) < c_2(\theta_P) < R$.

Proof. To show that $c_1(\theta_I) < c_2(\theta_P)$ use the characterization above and the concavity of the utility function. Since $\rho R > 1$, then $u'(c_1(\theta_I)) > u'(c_2(\theta_P))$, which by concavity gives $c_1(\theta_I) < c_2(\theta_P)$.

To show that $c_1(\theta_I) > 1$ and $c_2(\theta_P) < R$ we will appeal to the assumption that $-cu''(c)/u'(c) > 1$. Since $\rho < R$, we can write:

$$\begin{aligned}
\rho R u'(R) &< R u'(R) \\
&= 1 \cdot u'(1) + \int_1^R \frac{\partial}{\partial x} x u'(x) dx \\
&= u'(1) + \int_1^R \{u'(x) + x u''(x)\} dx \\
&= u'(1) + \int_1^R u'(x)\{1 + x u''(x)/u'(x)\} dx \\
&< u'(1)
\end{aligned}$$

But the above logic shows that the FOC is violated when we suppose to the contrary that $c_1(\theta_I) = 1$ and $c_2(\theta_P) = R$ since:

$$\frac{u'(c_1(\theta_I))}{u'(c_2(\theta_P))} = \frac{u'(1)}{u'(R)} > \rho R$$

From the resource constraint,

$$c_1(\theta_I) = (\lambda R)^{-1}\Big[R - (1-\lambda)c_2(\theta_P)\Big]$$

Which implies that the impatient agent's allocation is a decreasing function of the patient agent's allocation. Thus there exists some $c_1(\theta_I) > 1$ and $c_2(\theta_P) < R$ such that $u'(c_1(\theta_I)) = \rho R u'(c_2(\theta_P))$. □

Notice that the optimal allocation is Pareto efficient. That is, the allocation satisfies the incentive constraints, which turn out to be slack. Therefore informational frictions do not generate distortions in this economy.

More importantly, the efficient allocation achieves a greater degree of risk sharing relative to autarky. Whereas patient agents consume less, impatient agents will consume more relative to autarky. From an ex ante perspective this reallocation of resources from the patient to the impatient state is an improvement in expected utility for these risk averse agents. Therefore the cross-subsidization of liquidity risk improves welfare.

3.2 Atkeson and Lucas (RES 1992)

In this section I will first setup a static version of the Atkeson and Lucas (RES 1992) environment and highlight important tensions that arise due to private information. I will then consider a two period model and contrast the constrained efficient allocation with that in the static environment. I will then move to an infinite horizon model, provide simplifications and posit the recursive formulation of the problem. Given the recursive program I will then derive the immiseration result that in the infinite horizon, the distribution of consumption is degenerate with mass at the lowest level of consumption for all agents.

3.2.1 Static Economy with Two Types

Consider a one period economy, populated by a continuum of agents with period utility $\theta u(c)$, where $c \geq 0$ is consumption and θ is an *iid* random variable taking on values $\theta \in \Theta \equiv \{\theta_L, \theta_H\}$ with probabilities $\pi(\theta_L)$ and $\pi(\theta_H)$ respectively. Assume that θ is privately known by each agent. Assume that $u(\cdot)$ is concave, strictly increasing, twice continuously differentiable and satisfies Inada conditions. Lastly assume that each agent receives an endowment, e, of the consumption good.

A **mechanism** of this environment consists of agents' type-reporting strategies $(A_i)_{i\in[0,1]}$ and outcome function $c : A \rightarrow \mathbb{R}_+$. We will restrict attention to direct revelation mechanisms. To implement truthtelling as a Bayesian Nash equilibrium of the direct revelation mechanism, impose the following incentive constraints:

$$\theta_H u\Big(c(\theta_H)\Big) \geq \theta_H u\Big(c(\theta_L)\Big)$$

$$\theta_L u\Big(c(\theta_L)\Big) \geq \theta_L u\Big(c(\theta_H)\Big)$$

Before analyzing the static environment with private information, let's characterize the full information optimum as the solution to the following ex ante utilitarian social planner problem:

$$\max_{c(\theta_L),c(\theta_H)} \; \pi(\theta_H)\theta_H u\Big(c(\theta_H)\Big) + \pi(\theta_L)\theta_L u\Big(c(\theta_L)\Big)$$

$$\text{s.t.} \quad \textstyle\sum_\theta \pi(\theta)c(\theta) \quad \leq e \quad (RC, \lambda)$$

First order conditions are necessary and sufficient to characterize the optimum. FOCs are:

$$\pi(\theta_L)\theta_L u'\Big(c(\theta_L)\Big) = \pi(\theta_L)\lambda$$

$$\pi(\theta_H)\theta_H u'\Big(c(\theta_H)\Big) = \pi(\theta_H)\lambda$$

The FOCs and the concavity of $u(\cdot)$ imply that agents with higher marginal utility of consumption should receive a larger allocation:

$$\frac{u'\Big(c(\theta_H)\Big)}{u'\Big(c(\theta_L)\Big)} = \frac{\theta_L}{\theta_H} < 1 \qquad \Longrightarrow \qquad c(\theta_H) > c(\theta_L)$$

Notice that the Pareto efficient allocation satisfies the downward incentive constraint but violates the upward incentive constraint.

I will now characterize the constrained efficient allocation by solving the following ex ante social planner's problem that contains the full constraint set:

$$\begin{aligned}
\max_{c(\theta_L),c(\theta_H)} \; & \pi(\theta_H)\theta_H u\Big(c(\theta_H)\Big) + \pi(\theta_L)\theta_L u\Big(c(\theta_L)\Big) \\
\text{s.t.} \quad & \theta_H u\Big(c(\theta_H)\Big) \; \geq \theta_H u\Big(c(\theta_L)\Big) \quad (ICH, \pi(\theta_H)\mu) \\
& \theta_L u\Big(c(\theta_L)\Big) \; \geq \theta_L u\Big(c(\theta_H)\Big) \quad (ICL, \pi(\theta_L)\mu) \\
& \textstyle\sum_\theta \pi(\theta)c(\theta) \; \leq e \qquad\qquad\qquad (RC, \lambda)
\end{aligned}$$

It turns out that we can characterize the optimum solely by inspecting the incentive constraints. Notice that the shocks cancel out in each incentive constraint leaving that the constrainted efficient allocation must equate consumption, $c(\theta_H) = c(\theta_L)$, in order to satisfy both incentive constraints. The only allocation that equates consumption across states and satisfies the resource constraint is the agents' endowments. Therefore autarky is the constrained efficient allocation in this static economy with privately known marginal utilities.

The above results translate nearly exactly in an environment with $N \geq 3$ types of agents instead of $N = 2$ types.

3.2.2 Two Period Economy with Two Types

If the constrained Pareto efficient allocation is the autarkic allocation in a static environment, will dynamic incentives improve upon insurance?

To consider this question I will extend the static economy to two periods, $t = 0, 1$. Agents will now receive two taste shocks, $\{\theta_0, \theta_1\}$, such that $\theta_t \in \Theta \equiv \{\theta_L, \theta_H\}$ for each t. Each shock θ_t is iid across individuals and time. Agents receive an endowment e at each period. Agents have discount factor β and have access to a storage technology that returns the consumption good at rate $R > 1$. Assume for simplicity that $\beta = R^{-1}$. Agents have preferences:

$$U(c_0, c_1; \theta_0, \theta_1) = \theta_0 u(c_0) + \beta\theta_1 u(c_1)$$

where $u(\cdot)$ satisfies the same conditions as in the last section.

I again restrict attention to direct revelation mechanisms. The outcome functions are now history dependent:

$$c_0 : \Theta \to \mathbb{R}_+$$
$$c_1 : \Theta \times \Theta \to \mathbb{R}_+$$

Therefore, an allocation consists of consumption at both periods for each type of agent, $\{c_0(\theta_0), c_1(\theta_0, \theta_1)\}$. In order to enforce dynamic incentives I will impose two sets of incentive constraints.

I impose a set of second period incentive constraints that ensure agents reveal their type for any given first period shock,

$$\theta_1 u\Big(c_1(\theta_0, \theta_1)\Big) \geq \theta_1 u\Big(c_1(\theta_0, \widehat{\theta}_1)\Big) \qquad \forall \quad \theta_0, \theta_1, \widehat{\theta}_1$$

I then impose a set of first period incentive constraints that ensure agents reveal their types given their expected utility at period 2,

$$\theta_0 u\Big(c_0(\theta_0)\Big) + \beta \sum_{\theta_1} \pi(\theta_1)\theta_1 u\Big(c_1(\theta_0, \theta_1)\Big) \geq \theta_0 u\Big(c_0(\widehat{\theta}_0)\Big) + \beta \sum_{\theta_1} \pi(\theta_1)\theta_1 u\Big(c_1(\widehat{\theta}_0, \theta_1)\Big) \qquad \forall \quad \theta_0, \widehat{\theta}_0$$

Accordingly, we can write an ex ante utilitarian social planner's problem as:

$$\max_{c_0(\theta), c_1(\theta)} \sum_{\theta_0, \theta_1} \pi(\theta_0)\pi(\theta_1) \Big[\theta_0 u\Big(c(\theta_0)\Big) + \beta\theta_1 u\Big(c(\theta_1)\Big)\Big]$$

$$\begin{aligned}
\text{s.t.} \qquad \theta_L u\Big(c_1(\theta_0, \theta_L)\Big) &\geq \theta_L u\Big(c_1(\theta_0, \theta_H)\Big) \qquad \forall\, \theta_0 \\
\theta_H u\Big(c_1(\theta_0, \theta_H)\Big) &\geq \theta_H u\Big(c_1(\theta_0, \theta_L)\Big) \qquad \forall\, \theta_0 \\
\theta_0 u\Big(c_0(\theta_0)\Big) + \beta \sum_{\theta_1} \pi(\theta_1)\theta_1 u\Big(c_1(\theta_0, \theta_1)\Big) &\geq \theta_0 u\Big(c_0(\widehat{\theta}_0)\Big) + \beta \sum_{\theta_1} \pi(\theta_1)\theta_1 u\Big(c_1(\widehat{\theta}_0, \theta_1)\Big) \quad \forall\, \theta_0, \widehat{\theta}_0 \\
\sum_{\theta_0, \theta_1} \pi(\theta_0)\pi(\theta_1)\Big[c(\theta_0) + R^{-1}c(\theta_0, \theta_1)\Big] &\leq e + R^{-1}e
\end{aligned}$$

Before characterizing the constrained efficient allocation as the optimum of the above planner's problem, let's characterize the Pareto efficient allocation. Write the full information planner's problem as:

$$\max_{c_0(\theta),c_1(\theta)} \sum_{\theta_0,\theta_1} \pi(\theta_0)\pi(\theta_1)\left[\theta_0 u\Big(c(\theta_0)\Big) + \beta\theta_1 u\Big(c(\theta_1)\Big)\right]$$

$$\text{s.t.} \sum_{\theta_0,\theta_1} \pi(\theta_0)\pi(\theta_1)\Big[c(\theta_0) + R^{-1}c(\theta_0,\theta_1)\Big] \le e + R^{-1}e \quad (RC, \lambda)$$

First order conditions are necessary and sufficient:

$$\pi(\theta_0)\theta_0 u'\Big(c_0(\theta_0)\Big) = \pi(\theta_0)\lambda \qquad \forall\, \theta_0$$

$$\beta\pi(\theta_0)\pi(\theta_1)\theta_1 u'\Big(c_1(\theta_0,\theta_1)\Big) = R^{-1}\pi(\theta_0)\pi(\theta_1)\lambda \qquad \forall\, \theta_0,\theta_1$$

Thus we can characterize the optimum by using the FOCs as follows:

1. Combine the FOCs for first period consumption across types:

$$\theta_H u'\Big(c_0(\theta_H)\Big) = \theta_L u'\Big(c_0(\theta_L)\Big)$$

   By the concavity of $u(\cdot)$ and since $\theta_H/\theta_L > 1$, we find that $c_0(\theta_H) > c_0(\theta_L)$.

2. Combine the FOCs for second period consumption across $\theta_0 \in \{\theta_L, \theta_H\}$:

$$\theta_1 u'\Big(c_1(\theta_H,\theta_1)\Big) = \theta_1 u'\Big(c_1(\theta_L,\theta_1)\Big) \quad \forall\, \theta_1$$

   Therefore we have that second period consumption is invariant to first period shocks, $c_1(\theta_H,\theta_1) = c_1(\theta_L,\theta_1)$.

3. Combine the FOCs for second period consumption across both shocks $\theta_0, \theta_1 \in \{\theta_L, \theta_H\}$:

$$\theta_H u'\Big(c_1(\theta_0,\theta_H)\Big) = \theta_L u'\Big(c_1(\widehat{\theta}_0,\theta_L)\Big) \quad \forall\, \theta_0, \widehat{\theta}_0$$

   Therefore $c_1(\theta_0,\theta_H) > c_1(\widehat{\theta}_0,\theta_L)$.

4. Lastly, combine the FOCs intertemporally:

$$\frac{u'\Big(c_0(\theta_0)\Big)}{u'\Big(c_1(\theta_0,\theta_1)\Big)} = \frac{\theta_1}{\theta_0} \Longrightarrow \begin{cases} c_0(\theta_0) = c_1(\theta_0,\theta_1) & \text{for } \theta_0 = \theta_1 \\ c_0(\theta_L) < c_1(\theta_L,\theta_H) & \text{for } \theta_1 = \theta_H, \theta_0 = \theta_L \\ c_0(\theta_H) > c_1(\theta_H,\theta_L) & \text{for } \theta_1 = \theta_L, \theta_0 = \theta_H \end{cases}$$

By considering all four items of the characterization we obtain the result that given full information about agents, the social planner treats all agents of a type the same across time and gives high types greater consumption:

$$c_0(\theta_H) = c_1(\theta_L,\theta_H) = c_1(\theta_H,\theta_H) \;>\; c_1(\theta_L,\theta_L) = c_1(\theta_H,\theta_L) = c_0(\theta_L)$$

Next, notice that the Pareto efficient allocation will satisfy the second period downward incentive constraint but violate the second period upward incentive constraint. Accordingly, let's use the incentive constraints to characterize the constrained efficient allocation.

From the second period incentive constraints, we find $c_1(\theta_0, \theta_H) = c_1(\theta_0, \theta_L)$ for all $\theta_0 \in \Theta$:

$$\left.\begin{aligned} u\Big(c_1(\theta_0, \theta_H)\Big) &\geq u\Big(c(\theta_0, \theta_L)\Big) \\ u\Big(c_1(\theta_0, \theta_H)\Big) &\leq u\Big(c(\theta_0, \theta_L)\Big) \end{aligned}\right\} \implies c_1(\theta_0, \theta_H) = c(\theta_0, \theta_L) \quad \forall \theta_0 \in \Theta$$

Furthermore, by summing the upward and downward $first$ period incentive constraints and cancelling expectations terms, we find:

$$\theta_H u\Big(c_0(\theta_H)\Big) + \theta_L u\Big(c_0(\theta_L)\Big) \geq \theta_H u\Big(c_0(\theta_L)\Big) + \theta_L u\Big(c_0(\theta_H)\Big)$$

$$(\theta_H - \theta_L) u\Big(c_0(\theta_H)\Big) \geq (\theta_H - \theta_L) u\Big(c_0(\theta_L)\Big)$$

$$\implies \qquad c_0(\theta_H) \geq c_0(\theta_L)$$

Lastly, rewrite the the first period incentive constraints as and prove the subsequent claim:

$$(LUIC_0) \quad \beta \sum_{\theta_1} \pi(\theta_1)\theta_1 \Big[u\Big(c_1(\theta_L, \theta_1)\Big) - u\Big(c_1(\theta_H, \theta_1)\Big)\Big] \geq \theta_L \Big[u\Big(c_0(\theta_H)\Big) - u\Big(c_0(\theta_L)\Big)\Big]$$

$$(LDIC_0) \quad \beta \sum_{\theta_1} \pi(\theta_1)\theta_1 \Big[u\Big(c_1(\theta_L, \theta_1)\Big) - u\Big(c_1(\theta_H, \theta_1)\Big)\Big] \leq \theta_H \Big[u\Big(c_0(\theta_H)\Big) - u\Big(c_0(\theta_L)\Big)\Big]$$

Claim 3.2.1
The constrained efficient allocation must satisfy $c_1(\theta_L, \theta_1) \geq c_1(\theta_H, \theta_1)$ for all $\theta_1 \in \Theta$.

Proof. Suppose for contradiction that $c_1(\theta_L, \theta_1) < c_1(\theta_H, \theta_1)$. Then the expectations term on the LHS of the incentive constraints is negative. Then the $LDIC_0$ is satisfied given that $c_0(\theta_H) \geq c_0(\theta_L)$. However, the $LUIC_0$ is violated since the expectation term being negative implies that $c_0(\theta_H) < c_0(\theta_L)$. ☐

Notice that when $c_1(\theta_L, \theta_1) = c_1(\theta_H, \theta_1)$ for all $\theta_1 \in \Theta$, the first period incentive constraints tell us that $c_0(\theta_H) = c_0(\theta_L)$. If $c_1(\theta_L, \theta_1) > c_1(\theta_H, \theta_1)$ then in order to satisfy $LDIC_0$ we must have $c_0(\theta_H) > c_0(\theta_L)$.

Finally, in order to complete the characterization we will take FOCs of a relaxed social planner's problem. Since the incentive constraints showed us that only first period taste shocks matter for the constrained efficient consumption allocation, define:

$$\tilde{c}_1(\theta_0) \equiv c_1(\theta_0, \theta_1) \qquad \forall\, \theta_0, \theta_1 \in \Theta$$

In what follows I will be implicitly assuming that the second period incentive constraints hold with equality. Furthermore since $LDIC_0$ was slack, I guess and verify that $LUIC_0$ binds. Hence, consider the following relaxed ex ante utilitarian social planner's problem:

$$\max_{c_0(\theta), c_1(\theta)} \sum_{\theta_0, \theta_1} \pi(\theta_0)\pi(\theta_1) \Big[\theta_0 u\Big(c(\theta_0)\Big) + \beta\theta_1 u\Big(c(\theta_1)\Big)\Big]$$

$$\text{s.t.} \quad \beta \sum_{\theta_1} \pi(\theta_1)\theta_1 \Big[u\Big(\tilde{c}_1(\theta_L)\Big) - u\Big(\tilde{c}_1(\theta_H)\Big)\Big] \geq \theta_L \Big[u\Big(c_0(\theta_H)\Big) - u\Big(c_0(\theta_L)\Big)\Big] \quad (LUIC_0, \pi(\theta_L)\mu)$$

$$\sum_{\theta_0,\theta_1} \pi(\theta_0)\pi(\theta_1)\Big[c(\theta_0) + R^{-1}c(\theta_0,\theta_1)\Big] \leq e + R^{-1}e \quad (RC, \lambda)$$

The first order conditions are:

$$\pi(\theta_L)\theta_L u'\Big(c_0(\theta_L)\Big)(1+\mu) = \pi(\theta_L)\lambda$$

$$\pi(\theta_H)\theta_H u'\Big(c_0(\theta_H)\Big)\left(1 + \mu\frac{\pi(\theta_L)}{\pi(\theta_H)}\frac{\theta_L}{\theta_H}\right) = \pi(\theta_H)\lambda$$

$$\pi(\theta_L)\beta\sum_{\theta_1}\pi(\theta_1)\theta_1 \cdot u'\Big(\tilde{c}_1(\theta_L)\Big)(1+\mu) = \pi(\theta_L)R^{-1}\lambda$$

$$\pi(\theta_H)\beta\sum_{\theta_1}\pi(\theta_1)\theta_1 \cdot u'\Big(\tilde{c}_1(\theta_H)\Big)\left(1 + \mu\frac{\pi(\theta_L)}{\pi(\theta_H)}\right) = \pi(\theta_L)R^{-1}\lambda$$

By combining the last two FOCs (e.g. the $t = 1$ FOCs), noting that $\mu > 0$, and recalling that $\beta R = 1$ we obtain:

$$u'\Big(\tilde{c}_1(\theta_L)\Big) - u'\Big(\tilde{c}_1(\theta_H)\Big) = -\mu\Big[u'\Big(\tilde{c}_1(\theta_L)\Big) + \frac{\pi(\theta_L)}{\pi(\theta_H)}u'\Big(\tilde{c}_1(\theta_H)\Big)\Big] < 0$$

$$u'\Big(\tilde{c}_1(\theta_L)\Big) < u'\Big(\tilde{c}_1(\theta_H)\Big)$$

$$\tilde{c}_1(\theta_L) > \tilde{c}_1(\theta_H)$$

Furthermore, from the $LDIC_0$ the LHS is positive which requires that $c_0(\theta_H) > c_0(\theta_L)$.

The intuition for this characterization is as folllows. When an agent reports they have a high taste shock at $t = 0$, the planner is willing to give them insurance in order to accomodate the shock. However, in order to deter misreporting the social planner will use dynamic incentives as a punishment: by reporting a high shock at $t = 0$, the agent accepts strictly less consumption at $t = 1$ relative to having reported a low taste shock.

3.2.3 Infinite Horizon: Sequential Formulation

Encoding this history dependence into allocations can become computationally very expensive. In the previous section, having only considered a two period model, the characterization required some work. As such let's reformulate an infinite horizon sequential problem as a recursive program and apply standard dynamic programming results to characterize the constrained efficient allocation as decision rules of the dynamic program.

Suppose agents order consumption allocations according to period utility function $u(\cdot)$, which we will assume is concave, strictly increasing, twice continuously differentiable and satisfies Inada conditions. Denote the image of the utility function as $D \equiv u(\mathbb{R}_+)$. Next define the cost function as the inverse of the utility function as defined on D, $C(u) = u^{-1}$ which is a mapping $C : D \rightarrow \mathbb{R}_+$. Lastly, denote v as an agent's initial entitlement to expected discounted utility. Note that since lifetime discounted utility is obviously measured in utils, $v \in V \equiv D/(1-\beta)$. This "promise utility" is the key to writing the sequential problem as a dynamic program.

Continue to let shocks be *iid* across time and agents. Suppose that in each period the set of shocks is $\Theta = \{\theta_1, \ldots, \theta_N\}$, where $N \geq 2$ is finite and $\theta_1 < \theta_2 < \cdots < \theta_N$. Denote any history of shocks at time t as $\theta^t \equiv (\theta_0, \theta_1, \ldots, \theta_t)$, which is an element of the Cartesian product $\Theta^{t+1} \equiv \times_{i=0}^{t} \Theta$. Denote the probability that nature chooses $\theta_i \in \Theta$ as $\pi(\theta_i)$ and the probability that nature chooses a history θ^t as $\pi(\theta^t) \equiv \prod_{t=0}^{t} \pi(\theta_t)$. As a normalization, suppose that Θ satisfies $\sum_\theta \pi(\theta)\theta = 1$.

Agents report their type to a planner. A reporting strategy is given by $\sigma = \{\sigma(\theta^t)\}_{t=0}^{\infty}$ where $\sigma_t : \times_{i=0}^{t} \Theta \rightarrow \Theta$ for all t. We will restrict our attention to direct revelation mechanisms that implement truthtelling. A truthful report takes the form $\sigma_t(\theta^t) = \theta_t$. At each period the planner uses agents' reports to allocate consumption, $c_t : \times_{i=0}^{t} \Theta \times V \rightarrow \mathbb{R}_+$. An allocation is therefore a sequence of functions $\{c_t(\cdot|v)\}$. Then we may define a sequence of period utilities, $\{u_t(\cdot|v)\}$, where $u_t : \times_{i=0}^{t} \Theta \times V \rightarrow D$.

Lastly, define the distribution of initial promise utilities as ψ_0, and define a sequence of distributions as $\{\psi_t\}_{t=1}^{\infty}$. In words, each period the planner will promise a stream of expected discounted utility to agents, which implies a new distribution of promise utilities after each period. Therefore, an allocation and an initial distribution of promise utilities are sufficient to characterize the sequence of distributions. Denote the transition function for the distribution function by Ψ, such that

$$\psi_{t+1} = \Psi \psi_t \qquad \forall\, t$$

Given the above environment we can now define the constraint set for the planner. First, since we have restricted attention to direct revelation mechanisms we must include the appropriate incentive compatibility constraint. In this setting, the allocation $\{c_t(\cdot|v)\}$ must deliver higher ex ante lifetime utility to truthful reporting than to misreporting.

$$\sum_{t=0}^{\infty} \sum_{\theta^t \in \Theta^{t+1}} \beta^t \theta_t u\Big(c_t(\theta^t|v)\Big)\pi(\theta^t) \geq \sum_{t=0}^{\infty} \sum_{\theta^t \in \Theta^{t+1}} \beta^t \theta_t u\Big(c_t(\widehat{\theta}^t|v)\Big)\pi(\theta^t) \quad \forall\, t, \theta^t \in \Theta^{t+1}, v \ (IC)$$

Second, the planner must promise to deliver any agent's entitlement to expected lifetime utility. Hence the following promise keeping constraint.

$$v = \sum_{t=0}^{\infty} \sum_{\theta^t \in \Theta^{t+1}} \beta^t \theta_t u\Big(c_t(\theta^t|v)\Big)\pi(\theta^t) \quad (PKC)$$

Lastly, the allocation must be resource feasible at each period given an endowment stream $(e, e, \ldots)$.

$$\int_v \sum_{\theta^t \in \Theta^{t+1}} c_t(\theta^t|v)\pi(\theta^t) d\psi_t(v) \leq e \quad \forall\, t \quad (RC')$$

In order to show that the sequential problem (defined below) satisfies a Bellman equation, it will be useful to define the following time zero resource constraint.

$$\int_v \sum_{t=0}^{\infty} Q_t \sum_{\theta^t \in \Theta^{t+1}} c_t(\theta^t|v)\pi(\theta^t)d\psi_t(v) \le e \sum_{t=0}^{\infty} Q_t \quad (RC)$$

for given price sequence $\{Q_t\}$ such that $\sum_{t=0}^{\infty} Q_t < \infty$. Conveniently, Atkeson and Lucas show (Theorem 1) that we can construct prices so that any allocation that satisfies the time zero resource constraint will also satisfy the period-by-period resource constraint.

Atkeson and Lucas study an ex ante social planner's problem in which the social planner chooses an allocation to minimize resource use, e, subject to (IC), (PKC) and (RC). Note that the problem they solve is the dual to a social planner that maximizes the sum of utilities subject to the same constraint set. Since the resource constraint will bind, we can write the social planner's problem as:

$$S(\psi_0) = \min_{\{c_t\}} \int_v \sum_{t=0}^{\infty} Q_t \sum_{\theta^t \in \Theta^{t+1}} c_t(\theta^t|v)\pi(\theta^t)d\psi_t(v)$$
$$\text{s.t.} \quad (IC), (PKC)$$

3.2.4 Infinite Horizon: Recursive Formulation

Although I won't show it here, we could use standard theorems in Stokey, Lucas and Prescott (1989) to show that the sequential problem above admits a recursive representation. In particular, suppose that we study exponential prices of the form $Q_t = q^t$ for all t. Then minimization in the social planner's problem can be performed pointwise, e.g. for each $v \in V$. Then the sequential pointwise minimization problem for agent v is

$$K(v) \equiv \min_{\{c_t\}} \sum_{t=0}^{\infty} \sum_{\theta \in \Theta^{t+1}} q^t c_t(\theta^t|v)\pi(\theta^t)$$
$$\text{s.t.} \quad (IC), (PKC)$$

Then the Bellman equation is:

$$K(v) = \min_{u,w} \sum_{i=1}^{N} \pi(\theta_i)\Big[C\Big(u(\theta_i|v)\Big) + qK\Big(w(\theta_i|v)\Big)\Big]$$
$$\text{s.t.} \quad \theta_i u(\theta_i|v) + \beta w(\theta_i|v) \ge \theta_i u(\theta_j|v) + \beta w(\theta_j|v) \quad \forall\, i,j \quad (IC_{ij})$$
$$v = \textstyle\sum_{i=1}^{N} \pi(\theta_i)\Big[\theta_i u(\theta_i|v) + \beta w(\theta_i|v)\Big] \qquad (PKC)$$

where the function $w : \Theta \times D \to D$ denotes the expected discounted lifetime utility promised to the agent from tomorrow onward, and the function $u : \Theta \times D \to D$ denotes the period utility as a function of promised utility and contemporaneous report. In this formulation, pointwise optimization is performed given prices, q.

Interpret C as the amount of the consumption good needed to deliver a given utility level, as it was defined in the beginning of the section. In this recursive formulation of the problem, incentive constraints consider utility today, $u(\theta|v)$, and utility tomorrow, $w(\theta|v)$. The planner no longer writes contracts contingent on histories. In fact, as Green (1987) shows, this form of incentive compatibility implies the full incentive compatibility that was required in the sequential planner's problem. Lastly, the social planner will provide allocations that satisfy the new promise keeping constraint, which requires the planner to choose allocation rules (u, w) to deliver an expected utility of v.

An interpretation for the pointwise planner problem is that there is a continuum of "component planners" who write contracts only with agents who have promise utility v, for each $v \in V$. The component planner is an ex ante utilitarian social planner who chooses (u, w) to minimize resources required to deliver type-v agents' promise utility, subject to incentive compatibility. There is a component planner for each v, and accordingly component planners contract with each other through future entitlements $w(\theta|v)$.

The Bellman has several important properties.

Claim 3.2.2
The value function $K(\cdot)$ is strictly convex, strictly increasing and differentiable. Furthermore, if $\underline{v} \equiv \inf V$ then $\lim_{v \to \underline{v}} K'(v) = 0$.

Proof. See Stokey, Lucas and Prescott chapter 4 for proofs of strict convexity, strict monotonicity and differentiabilty. The limit of the derivative follows from the *strict* monotonicity and convexity. □

Before proceeding with the characterization of the component planner's problem, let's first consider the full information optimum by solving a component planner's problem subject only to the promise keeping constraint. The following two equations characterize the optimum:

$$u(\theta_i|v) = C'^{-1}\Big(\theta_i K'(v)\Big)$$

$$K'\Big(w(\theta_i|v)\Big) = \frac{\beta}{q} K'(v)$$

The first equation tells us that agents with higher marginal utilities of consumption (e.g. higher taste shocks) will be given more consumption. This follows from the strict concavity of the utility function and the definition of the inverse utility function. The second equation pins down the future utility, w. Notice that if $q = \beta$, then $v = w(\theta_i|v)$ for all θ_i and then all initial inequality is permanently transmitted into the future.

Before solving the component planner's problem under private information, I will propose several reductions of the constraint set through a series of claims and proofs. Using the reduced constraint set, I will then propose a "relaxed" component planner's problem and characterize the optimal mechanism of the environment. I will conclude this section with a discussion of the immiseration result, that optimal mechanisms have a long run property of promising the lowest level of consumption to all but a measure zero fraction of the population. That is, the long run distribution of promise utilities is degenerate with minimum consumption, $\lim_{t \to \infty} \psi_t(\{\inf V\}) = 1$.

Claim 3.2.3 (Monotonicity)
For each $i > j$, where $\theta_i > \theta_j$, we have (i) $u(\theta_i|v) \geq u(\theta_j|v)$ and (ii) $w(\theta_i|v) \leq w(\theta_j|v)$.

Proof. To prove (i), take some $i, j \in \{1, 2, \ldots, N\}$ such that $i > j$ and sum the downward incentive constraint (type i reports he is type j) and the upward incentive constraint (type j reports he is type i). Summing the IC's gives:

$$\theta_i u(\theta|v) + \theta_j u(\theta_j|v) \geq \theta_i u(\theta_j|v) + \theta_j u(\theta_i|v) \quad \Longrightarrow \quad u(\theta_i|v) \geq u(\theta_j|v)$$

Therefore, $u(\theta_i|v) \geq u(\theta_j|v)$ for all $\theta_i > \theta_j$.

To prove (ii), suppose to the contrary that $w(\theta_i|v) > w(\theta_j|v)$. But then the upward incentive constraint is violated. To see this, write the upward IC as:

$$\theta_j \Big[u(\theta_j|v) - u(\theta_i|v) \Big] \geq \beta \Big[w(\theta_i|v) - w(\theta_j|v) \Big]$$

From part (i), we know that the LHS is nonpositive. From our supposition, the RHS must be strictly positive. But then the LHS must be strictly positive which contradicts the LHS being nonpositive. Therefore $w(\theta_i|v) \leq w(\theta_j|v)$ as desired. □

Next I will prove that local incentive constraints are sufficient to characterize global incentive compatibility. For ease, I will abreviate "local downward incentive constraint for type i" by LDIC$_i$ and "local upward incentive constraint for type i" by LUIC$_i$.

Claim 3.2.4 (Local and Global Downward ICs)
If LDIC$_i$ is satisfied for each i, then for each i and $j < i$, LDIC$_i$ implies

$$\theta_i u(\theta_i|v) + \beta w(\theta_i|v) \geq \theta_i u(\theta_j|v) + \beta w(\theta_j|v)$$

Proof. Suppose LDIC$_i$ and LDIC$_{i+1}$ both hold. Then from LDIC$_i$ we can write:

$$\theta_{i+1} u(\theta_i|v) + \beta w(\theta_i|v) \geq \theta_{i+1} u(\theta_{i-1}|v) + \beta w(\theta_{i-1}|v)$$

The above IC combined with LDIC$_{i+1}$ gives:

$$\left.\begin{aligned} \theta_{i+1} u(\theta_{i+1}|v) + \beta w(\theta_{i+1}|v) &\geq \theta_{i+1} u(\theta_i|v) + \beta w(\theta_i|v) \\ \theta_{i+1} u(\theta_i|v) + \beta w(\theta_i|v) &\geq \theta_{i+1} u(\theta_{i-1}|v) + \beta w(\theta_{i-1}|v) \end{aligned}\right\} \Longrightarrow \begin{gathered} \theta_{i+1} u(\theta_{i+1}|v) + \beta w(\theta_{i+1}|v) \geq \\ \theta_{i+1} u(\theta_{i-1}|v) + \beta w(\theta_{i-1}|v) \end{gathered}$$

If LDIC$_{i-1}$ holds, then we can use the same procedure to show that:

$$\theta_{i+1} u(\theta_{i+1}|v) + \beta w(\theta_{i+1}|v) \geq \theta_{i+1} u(\theta_{i-2}|v) + \beta w(\theta_{i-2}|v)$$

And therefore we can obtain the result through iteration. □

Claim 3.2.5 (Local and Global Upward ICs)
If LUIC$_i$ is satisfied for each i, then for each i and $j > i$, LUIC$_i$ implies

$$\theta_i u(\theta_i|v) + \beta w(\theta_i|v) \geq \theta_i u(\theta_j|v) + \beta w(\theta_j|v)$$

Proof. Suppose LUIC$_i$ and LUIC$_{i+1}$ both hold. Then from LUIC$_{i+1}$ we can write:

$$\theta_i u(\theta_{i+1}|v) + \beta w(\theta_{i+1}|v) \geq \theta_i u(\theta_{i+2}|v) + \beta w(\theta_{i+2}|v)$$

The above IC combined with LUIC$_i$ gives:

$$\left.\begin{aligned} \theta_i u(\theta_i|v) + \beta w(\theta_i|v) &\geq \theta_i u(\theta_{i+1}|v) + \beta w(\theta_{i+1}|v) \\ \theta_i u(\theta_{i+1}|v) + \beta w(\theta_{i+1}|v) &\geq \theta_i u(\theta_{i+2}|v) + \beta w(\theta_{i+2}|v) \end{aligned}\right\} \Longrightarrow \theta_i u(\theta_i|v) + \beta w(\theta_i|v) \geq \theta_i u(\theta_{i+2}|v) + \beta w(\theta_{i+2}|v)$$

And therefore we can obtain the result through iteration. □

The previous two results allow us to restrict the set of incentive compatibility constraints to only local ICs. In the next two results I will show that only upward local ICs will bind, thereby restricting the set of ICs to upward local ICs.

Claim 3.2.6 (LUICs Bind)
LUICs are binding for each type i.

Proof. Proceed by contradiction. Suppose LUIC$_i$ is slack for some type i:

$$\theta_i u(\theta_i|v) + \beta w(\theta_i|v) > \theta_i u(\theta_{i+1}|v) + \beta w(\theta_{i+1}|v)$$

But then the planner can decrease $w(\theta_i|v)$ by some $\varepsilon_i > 0$ until LUIC$_i$ binds. Denote $\widehat{w}(\theta_i|v) = w(\theta_i|v) - \varepsilon_i$ and write the new LUIC$_i$ as:

$$\theta_i u(\theta_i|v) + \beta \widehat{w}(\theta_i|v) = \theta_i u(\theta_{i+1}|v) + \beta w(\theta_{i+1}|v)$$

However, decreasing $w(\theta_i|v)$ may cause $LUIC_{i-1}$ to become slack, in which case we must choose some $\varepsilon_{i-1} > 0$ in order to construct $\widehat{w}(\theta_{i-1}|v) = w(\theta_{i-1}|v) - \varepsilon_{i-1}$ that ensures LUIC$_{i-1}$ binds:

$$\theta_{i-1} u(\theta_{i-1}|v) + \beta w(\theta_{i-1}|v) > \theta_{i-1} u(\theta_{i-1}|v) + \beta \widehat{w}(\theta_{i-1}|v) = \theta_i u(\theta_i|v) + \beta \widehat{w}(\theta_i|v)$$

Thus LUIC$_i$ binds.

We can iteratively continue to decrease each type's promise utility until we construct a sequence of functions $\{\widehat{w}(\theta_i|v)\}_{i=1}^{N-1}$ for which $LUIC_i$ binds. Lastly, this new promised utility contract is resource feasible since (i) current period utility has not changed and (ii) decreasing future utility makes delivering future consumption more affordable and thus resource feasible. □

Claim 3.2.7 (LDICs are Slack)
LDICs are slack for each type i.

Proof. Proceed by contradiction. Suppose LDIC$_i$ is binding for some i. Then sum LDIC$_i$ and LUIC$_{i-1}$, which are both binding due to our assumption and the above claim 3.2.6.

$$\theta_i u(\theta_i|v) + \beta w(\theta_i|v) = \theta_i u(\theta_{i-1}|v) + \beta w(\theta_{i-1}|v)$$
$$\theta_{i-1} u(\theta_{i-1}|v) + \beta w(\theta_{i-1}|v) = \theta_{i-1} u(\theta_i|v) + \beta w(\theta_i|v)$$

Summation of the two LUICs yields:

$$\theta_i \Big[u(\theta_i|v) - u(\theta_{i-1}|v)\Big] = \theta_{i-1} \Big[u(\theta_i|v) - u(\theta_{i-1}|v)\Big] \quad \Longrightarrow \quad \theta_i = \theta_{i-1}$$

But this last line is a clear contradiction, since $\theta_i > \theta_{i-1}$. □

Now that we have established that upward local incentive constraints are sufficient to characterize the constraint set, consider a "relaxed" program:

$$K(v) = \min_{u,w} \sum_{i=1}^{N} \pi(\theta_i)\Big[C\Big(u(\theta_i|v)\Big) + qK\Big(w(\theta_i|v)\Big)\Big]$$

$$\text{s.t.} \quad \theta_i u(\theta_i|v) + \beta w(\theta_i|v) \geq \theta_i u(\theta_{i+1}|v) + \beta w(\theta_{i+1}|v) \quad \forall\, i < N \quad (LUIC_i, \mu_i)$$

$$v = \textstyle\sum_{i=1}^{N} \pi(\theta_i)\Big[\theta_i u(\theta_i|v) + \beta w(\theta_i|v)\Big] \qquad (PKC, \lambda)$$

For all $i \in \{2, \ldots, N-1\}$ the first order conditions for this program are:

$$\pi(\theta_i)\Big[C'\Big(u(\theta_i|v)\Big) - \lambda\theta_i\Big] = \mu_i\theta_i - \mu_{i-1}\theta_{i-1}$$

$$\pi(\theta_i)\Big[qK'\Big(w(\theta_i|v)\Big) - \beta\lambda\Big] = \beta\mu_i - \beta\mu_{i-1}$$

For $i \in \{1, N\}$, the FOCs with respect to $w(\theta|v)$ are:

$$\pi(\theta_1)\Big[qK'\Big(w(\theta_1|v)\Big) - \beta\lambda\Big] = \beta\mu_1$$

$$\pi(\theta_N)\Big[qK'\Big(w(\theta_N|v)\Big) - \beta\lambda\Big] = -\beta\mu_{N-1}$$

And the envelope condition gives:

$$K'(v) = \lambda$$

I will now use the FOCs and EC to prove a series of claims that will culminate in the immiseration result.

Claim 3.2.8
Define $v_{t+1}(\theta^t) = w(\theta_t|v_t(\theta^{t-1}))$. The stochastic process for $\{K'(v_t)\}$ is a super martingale.

Proof. Substituting the envelope condition into the FOC with respect to $w(\theta|v)$ and summing over θ_i we obtain:

$$\sum_{i=1}^{N} \pi(\theta_i)K'\Big(w(\theta_i|v)\Big) - \frac{\beta}{q}K'(v) = \frac{\beta}{q}\sum_{i=1}^{N}(\mu_i - \mu_{i-1})$$

where $\mu_N = \mu_0 = 0$. Therefore we can rewrite the condition as:

$$\sum_{i=1}^{N} \pi(\theta_i)K'\Big(w(\theta_i|v)\Big) = \frac{\beta}{q}K'(v) \tag{3}$$

Notice that if $q = \beta$, then the stochastic process for $\{K'(v_t)\}$ is a martingale. If $q > \beta$ then the process is a super martingale. Therefore:

$$\sum_{i=1}^{N} \pi(\theta_i)K'\Big(w(\theta_i|v)\Big) \leq K'(v)$$

□

Since $K'(v)$ is a super martingale we can prove that the planner generates incentives by increasing the spread of promise utility.

Claim 3.2.9 (Spreading Property)
For all $v \in V$, we have $w(\theta_N|v) < v < w(\theta_1|v)$.

Proof. I will prove that $w(\theta_N|v) < v$. The other inequality follows by a symmetric argument.

Suppose for contradiction that $w(\theta_N|v) \geq v$. Since $K(v)$ is strictly convex and since $v \leq w(\theta_N|v) \leq \cdots \leq w(\theta_1|v)$ we know that $K'(v) \leq K'(w(\theta_N|v)) \leq \cdots \leq K'(w(\theta_1|v))$. Further note that since $q \geq \beta$ we know that $\beta/qK'(v) \leq K'(v)$. Therefore from equation (3) we must have that

$$\frac{\beta}{q}K'(v) = K'\Big(w(\theta_N|v)\Big) = \cdots = K'\Big(w(\theta_1|v)\Big) \quad \Longrightarrow \quad w(\theta_i|v) = w(\theta_{i+1}|v)\ \forall i < N$$

If these equalities did not hold then the LHS of (3) would be strictly greater than the RHS.

Next substitute λ from the envelope condition into the FOCs with respect to $w(\theta|v)$. This yields for each i:

$$\pi(\theta_i)\Big[qK'\Big(w(\theta_i|v)\Big) - \beta K'(v)\Big] = \beta\mu_i - \beta\mu_{i-1}$$

where for $i \in \{1, N\}$ we can define $\mu_i = 0$ WLOG. Since $\frac{\beta}{q}K'(v) = K'(w(\theta_i|v))$ for all i, the LHS is equal to zero. Then the above condition implies that $\mu_1 = \mu_N = \mu_2 = \mu_{N-1} = 0$. But then by iteratively checking the condition we find that $\mu_i = 0$ for each i.

Then from the FOC with respect to $u(\theta|v)$ we find that

$$\pi(\theta_i)\Big[C'\Big(u(\theta_i|v)\Big) - \lambda\theta_i\Big] = 0 \quad \Longrightarrow \quad \frac{1}{\theta_i}C'\Big(u(\theta_i|v)\Big)$$

From the convexity of $C(\cdot)$ the above implies $u(\theta_1|v) < u(\theta_2|v) < \cdots < u(\theta_N|v)$. But from the incentive constraints if $w(\theta_i|v) = w(\theta_{i+1}|v)$ for each $i < N$ then requires $u(\theta_i|v) = u(\theta_{i+1}|v)$ for each $i < N$. This contradicts $u(\theta_1|v) < u(\theta_2|v) < \cdots < u(\theta_N|v)$. □

Now we are in a position to show the immiseration result.

Claim 3.2.10 (Immiseration)
As $t \to \infty$, a measure zero of agents receive consumption above the lowest possible consumption stream. That is $\lim_{t\to\infty} c_t(\theta|v_t(\theta^{t-1})) = 0$ and $\psi(\{\inf V\}) = 1$ almost surely.

Proof. From claim 3.2.2 we know that the value function is strictly increasing and therefore $K'(v) \geq 0$ for all $v \in V$. A corollary to Doob's second Martingale Convergence Theorem states that the process $\{K'(v_t)\}$ converges almost surely to a random variable. I will first show that $\lim_{t\to\infty} K'(v_t) = 0$ almost surely.

Suppose for contradiction that $\lim_{t\to\infty} K'(v_t) = \xi > 0$ almost surely for some $\xi \in (0, \infty)$. Then we must have $\lim_{t\to\infty} v_t = \overline{v} < \infty$ almost surely. Therefore in the limit the planner allocates future utility $w(\theta_i|\overline{v}) = \overline{v}$ for each i. But $\forall\, v$ the Spreading Property holds, $w(\theta_N|v) < v < w(\theta_1|v)$. Hence $w(\theta_i|\overline{v}) = \overline{v}$ violates optimality.

Thus, $\lim_{t\to\infty} K'(v_t) = 0$ requires that $\lim_{t\to\infty} v_t = \inf V$ almost surely. This means that spreading is not violated only if immiseration occurs. Lastly, this means that $c(\theta|\{\inf V\}) = 0$ for all θ. Therefore the distribution of future utility places unit mass on misery: $\psi(\{\inf V\}) = 1$ almost surely. □

On a technical note, the immiseration result is equivalent to the non-existence of a steady state distribution for ψ. It shows that the transition function Ψ has a downward drift as seen in equation (3). The key features for the result are the downward drift in incentive provision, the spreading of promise utilities as a result of incentive provision, and the monotonicity of the Bellman equation. These features imply that the planner provides incentives by pushing off consumption to the future and thereby drives contemporaneous consumption to zero.

3.2.5 Example: Log Utility

To be completed.

3.3 Farhi and Werning (JPE 2007)

Farhi and Werning (JPE 2007) take Atkeson and Lucas' environment and make a small change. They assume that the social planner discounts the future at a lower rate than private agents. That is they assume the planner's discount factor is $\hat{\beta} > \beta$. It turns out that this small change in the environment admits a stationary distribution over promise utilities, ψ, which is strictly bounded away from misery: $\psi(\{\inf V\}) = 0$ almost surely. This suggests that social mobility is possible unlike Atkeson and Lucas' economy.

For simplicity I will make the additional assumption that the period utility function, $u(\cdot)$, is unbounded above and below. This assumption is not necessary for the main results. This assumption allows us to characterize the set of promise utilities as $V = \mathbb{R}$.

Accordingly take as given the entire setup of the environment from section 3.2.3. However instead of studying the dual to the social planner's problem as Atkeson and Lucas, Farhi and Werning study the primal problem. We can write the ex ante utility social planner's problem as:

$$S(\psi_0) = \min_{\{c_t\}} \int_v \sum_{t=0}^{\infty} \sum_{\theta^t \in \Theta^{t+1}} \hat{\beta}^t \theta_t u\Big(c_t(\theta^t|v)\Big)\pi(\theta^t) d\psi_t(v)$$

s.t.

$$\sum_{t=0}^{\infty}\sum_{\theta^t\in\Theta^{t+1}}\beta^t\theta_t\left[u\Big(c_t(\theta^t|v)\Big)-u\Big(c_t(\widehat{\theta}^t|v)\Big)\right]\pi(\theta^t)\geq 0\quad\forall\, t,\theta^t\in\Theta^{t+1},v$$

$$v=\sum_{t=0}^{\infty}\sum_{\theta^t\in\Theta^{t+1}}\beta^t\theta_t u\Big(c_t(\theta^t|v)\Big)\pi(\theta^t)$$

$$\int_v\sum_{t=0}^{\infty}Q_t\sum_{\theta^t\in\Theta^{t+1}}c_t(\theta^t|v)\pi(\theta^t)d\psi_t(v)\leq e\sum_{t=0}^{\infty}Q_t$$

Farhi and Werning study exponential prices, as before, of the form $Q_t = q^t$. However, since they can obtain a stationary distribution over promise utilities they argue that the only exponential prices that are consistent with stationarity take the form $q = \hat{\beta}$. Suppose that instead $q > \hat{\beta}$ or $q < \hat{\beta}$. Then consumption streams would drift downward or upward, respectively. Given that $q = \hat{\beta}$, we can rewrite the social planner's problem by defining η as the lagrange multiplier on the resource constraint and putting the resource constraint directly in the objective function. Then the ex ante utilitarian social planner's problem takes the form:

$$S(\psi_0)=\max_{\{c_t\}}\int_v\sum_{t=0}^{\infty}\sum_{\theta^t\in\Theta^{t+1}}\hat{\beta}^t\left[\theta_t u\Big(c_t(\theta^t|v)\Big)-\eta C\Big(u\Big(c_t(\theta^t|v)\Big)\Big)\right]\pi(\theta^t)d\psi_t(v)$$

$$\text{s.t.}\quad\sum_{t=0}^{\infty}\sum_{\theta^t\in\Theta^{t+1}}\beta^t\theta_t\left[u\Big(c_t(\theta^t|v)\Big)-u\Big(c_t(\widehat{\theta}^t|v)\Big)\right]\pi(\theta^t)\geq 0\quad\forall\, t,\theta^t\in\Theta^{t+1},v$$

$$v=\sum_{t=0}^{\infty}\sum_{\theta^t\in\Theta^{t+1}}\beta^t\theta_t u\Big(c_t(\theta^t|v)\Big)\pi(\theta^t)$$

3.3.1 Recursive Formulation

As before we will solve the social planner's problem with pointwise optimization, e.g. for each $v \in V$. Then the sequential pointwise maximization problem for agent v is:

$$K(v)\equiv\max_{\{c_t\}}\sum_{t=0}^{\infty}\sum_{\theta\in\Theta^{t+1}}\hat{\beta}^t\left[\theta_t u\Big(c_t(\theta^t|v)\Big)-\eta C\Big(u\Big(c_t(\theta^t|v)\Big)\Big)\right]\pi(\theta^t)$$

$$\text{s.t.}\quad (IC),(PKC)$$

Note that since the constraint set has not changed from the Atkeson and Lucas component planner problem, the "relaxed" constraint set still characterizes the constraint set in this setting. Hence the Bellman is:

$$K(v)=\max_{u(\theta|v),w(\theta|v)}\sum_{i=1}^{N}\pi(\theta_i)\Big[\theta_i u(\theta_i|v)-\eta C\Big(u(\theta_i|v)\Big)+\hat{\beta}K(w(\theta_i|v))\Big]$$

$$\text{s.t.}\quad\theta_i u(\theta_i|v)+\beta w(\theta_i|v)\geq\theta_i u(\theta_{i+1}|v)+\beta w(\theta_{i+1}|v)\quad\forall\, i<N\quad(LUIC_i,\mu_i)$$

$$v=\textstyle\sum_{i=1}^{N}\pi(\theta_i)\Big[\theta_i u(\theta_i|v)+\beta w(\theta_i|v)\Big]\qquad(PKC,\lambda)$$

The Bellman has several important properties.

Claim 3.3.1
The value function $K(\cdot)$ is strictly concave and differentiable.

Proof. See Stokey, Lucas and Prescott chapter 4 for proofs. □

These properties allow us to characterize the shape of the Bellman.

Claim 3.3.2
The Bellman satisfies the following two properties:

(i) $\lim_{v \to -\infty} K(v) = \lim_{v \to +\infty} K(v) = \lim_{v \to +\infty} K'(v) = -\infty$

(ii) $\lim_{v \to -\infty} K'(v) = 1$

Proof. First let's define several objects. First define the highest value that the objective function can take at any given period, t:

$$m \equiv \max_{c \geq 0, \theta \in \Theta} \Big\{ \theta u(c) - \eta c \Big\}$$

Next define the value function for the full information problem as a function of promise utility and discount factor:

$$h(v; \hat{\beta}) \equiv \max_{\{u\}} \sum_{t=0}^{\infty} \sum_{\theta \in \Theta^{t+1}} \hat{\beta}^t \pi(\theta^t) \Big[\theta_t u\Big(\theta^t | v\Big) - \eta C\Big(u(\theta^t | v)\Big) \Big]$$

$$\text{s.t.} \quad v = \sum_{t=0}^{\infty} \sum_{\theta \in \Theta^{t+1}} \hat{\beta}^t \pi(\theta^t) \theta_t u\Big(\theta^t | v\Big)$$

Because the full information program has no incentive constraint we know $K(v) \leq h(v; \hat{\beta})$ for all $v \in V$. For notational simplicity define the optimal sequence of utility that solves the full information problem as $\{u_{\hat{\beta}}(\theta^t)\}_t$ and denote the optimal value of the objective function at period t as $\varphi_{\hat{\beta}}(\theta^t)$.

It will be convenient to define the constraint set under the full information problem as:

$$\Gamma(\hat{\beta}) = \left\{ \{u\} \;\middle|\; v = \sum_{t=0}^{\infty} \sum_{\theta \in \Theta^{t+1}} \hat{\beta}^t \pi(\theta^t) \theta_t u\Big(\theta^t | v\Big) \right\}$$

Notice that when $\hat{\beta} = \beta$, we can write the full information problem as:

$$h(v; \beta) = \max_{u \in \Gamma(\beta)} \sum_{t=0}^{\infty} \sum_{\theta \in \Theta^{t+1}} \beta^t \pi(\theta^t) \Big[\theta_t u\Big(\theta^t | v\Big) - \eta C\Big(u(\theta^t | v)\Big) \Big]$$

$$= v - \min_{u \in \Gamma(\beta)} \sum_{t=0}^{\infty} \sum_{\theta \in \Theta^{t+1}} \hat{\beta}^t \pi(\theta^t) \eta C\Big(u(\theta^t)\Big)$$

$$\equiv v - \eta F(v)$$

Notice that $F(v)$ defines a standard programming problem, in which $F(v)$ is strictly convex and $\lim_{v\to-\infty} F(v) = \lim_{v\to+\infty} F(v) = +\infty$.

Lastly, define the optimal sequence of utility that solves the full information problem with $\hat{\beta} = \beta$ as $\{u_\beta(\theta^t)\}_t$ and denote the optimal value of the objective function at period t as $\varphi_\beta(\theta^t)$.

Note that $\varphi_\beta(\theta^t) \leq m$ and $\varphi_{\hat{\beta}}(\theta^t) \leq m$ for all $\theta^t \in \Theta^{t+1}$.

Upper Bound

Having defined these objects, consider the following derivation of an upper bound on $h(v;\hat{\beta})$:

$$\begin{aligned}
h(v;\hat{\beta}) &= \sum_{t=0}^{\infty} \hat{\beta}^t \sum_{\theta\in\Theta^{t+1}} \pi(\theta^t)\varphi_{\hat{\beta}}(\theta^t) + \sum_{t=0}^{\infty} \beta^t \sum_{\theta\in\Theta^{t+1}} \pi(\theta^t)\varphi_\beta(\theta^t) - \sum_{t=0}^{\infty} \beta^t \sum_{\theta\in\Theta^{t+1}} \pi(\theta^t)\varphi_\beta(\theta^t) \\
&= \sum_{t=0}^{\infty} \hat{\beta}^t \sum_{\theta\in\Theta^{t+1}} \pi(\theta^t)\varphi_{\hat{\beta}}(\theta^t) \qquad + v - \eta F(v) \qquad - \sum_{t=0}^{\infty} \beta^t \sum_{\theta\in\Theta^{t+1}} \pi(\theta^t)\varphi_\beta(\theta^t) \\
&\leq \sum_{t=0}^{\infty} \hat{\beta}^t m + v - \eta F(v) - \sum_{t=0}^{\infty} \beta^t m \\
&\leq v - \eta F(v) + m \\
&= h(v;\beta) + m \\
&\equiv K_{max}(v)
\end{aligned}$$

Therefore we can now write, $K(v) \leq h(v;\hat{\beta}) \leq K_{max}(v)$ for all $v \in V$. Since we know that $F(v)$ is strictly convex with limits described above, we know that

$$\lim_{v\to-\infty} K_{max}(v) = \lim_{v\to+\infty} K_{max}(v) = -\infty$$

But since $K(v) \leq K_{max}(v)$ for all $v \in V$, this means that

$$\lim_{v\to-\infty} K(v) = \lim_{v\to+\infty} K(v) \leq \lim_{v\to-\infty} K_{max}(v) = \lim_{v\to+\infty} K_{max}(v) = -\infty$$

Hence $\lim_{v\to-\infty} K(v) = \lim_{v\to+\infty} K(v) = -\infty$.

Next we want to show that $\lim_{v\to-\infty} K'(v) = 1$ and that $\lim_{v\to+\infty} K'(v) = -\infty$. To do so let's characterize the derivative of $K_{max}(v)$:

$$K'_{max}(v) = 1 - F'(v)$$

Since $F(v)$ is strictly convex, $F'(v) > 0$ for all $v \in V$. Therefore

$$\lim_{v\to-\infty} F'(v) = 0 \qquad \text{and} \qquad \lim_{v\to+\infty} F'(v) = +\infty$$

Since $K(v) \leq K_{max}(v)$ for all $v \in V$, and since both $K(v)$ and $K_{max}(v)$ are strictly concave we obtain:

$$\lim_{v \to +\infty} K'(v) \leq \lim_{v \to +\infty} K'_{max}(v) = -\infty$$

$$\lim_{v \to -\infty} K'(v) \leq \lim_{v \to -\infty} K'_{max}(v) = 1$$

Since K_{max} is an upper bound on $K(v)$, we know that $K(v) \to -\infty$ as $v \to +\infty$.

<u>Lower Bound</u>

In order to further characterize the limit of $K(v)$ as $v \to -\infty$, we will construct a lower bound on $K(v)$. To do so, define the following allocations as functions of the optimal allocations under the private information problem with entitlement v_0:

$$\widetilde{u}(\theta, v) = u(\theta|v_0) + (v - v_0) \qquad \text{and} \qquad \widetilde{w}(\theta, v) = w(\theta|v_0)$$

Fix $v_0 \in intV$. Define the following function, $W(\cdot;\cdot)$, as the value function derived from these allocations:

$$\begin{aligned} K_{min}(v; v_0) &= \sum_{i=1}^{N} \pi(\theta_i)\Big[\theta_i \widetilde{u}(\theta, v) - \eta C\Big(\widetilde{u}(\theta, v)\Big) + \hat{\beta} K\Big(\widetilde{w}(\theta, v)\Big)\Big] \\ &= \sum_{i=1}^{N} \pi(\theta_i)\Big[\theta_i\Big(u(\theta|v_0) + (v - v_0)\Big) - \eta C\Big(u(\theta|v_0) + (v - v_0)\Big) + \hat{\beta} K\Big(w(\theta|v_0)\Big)\Big] \end{aligned}$$

We can apply Benveniste and Scheinkman's theorem since

(a) $-C(\cdot)$ is strictly concave and continuously differentiable and therefore so is $K_{min}(v; v_0)$.

(b) Since if $v \neq v_0$ we know

$$\begin{aligned} \widetilde{u}(\theta, v) &= u(\theta|v_0) + (v - v_0) & &\neq u(\theta|v) \\ \widetilde{w}(\theta, v) &= w(\theta|v_0) & &\neq w(\theta|v) \end{aligned}$$

And therefore the following only holds with equality when $v = v_0$:

$$K_{min}(v; v_0) \leq \max_{u,w} \sum_{i=1}^{N} \pi(\theta_i)\Big[\theta_i u(\theta|v) - \eta C\Big(u(\theta|v)\Big) + \hat{\beta} K\Big(w(\theta|v)\Big)\Big] \; = \; K(v)$$
$$s.t. \; (PKC), (IC)$$

Thus we apply the Benveniste-Scheinkman theorem to obtain

$$\begin{aligned} K'(v_0) = K'_{min}(v_0, v_0) &= \sum_{i=1}^{N} \pi(\theta_i)\Big[\theta_i - \eta C'\Big(\widetilde{u}(\theta, v_0)\Big)\Big] \\ &= 1 - \eta \sum_{i=1}^{N} \pi(\theta_i) C'\Big(u(\theta|v_0)\Big) \end{aligned}$$

where by assumption $\sum_i \theta_i = 1$. Since $C(\cdot)$ is strictly convex, e.g. $C'(u) \geq D \forall u \in D$, then $K'(v) \leq 1$ for all $v \in V$.

Lastly, since for all $v_0 \in IntV$,

$$K'_{min}(v,;v_0) = \sum_{i=1}^{N} \pi(\theta_i)\Big[\theta_i - \eta C'\Big(u(\theta|v_0) + (v - v_0)\Big)\Big]$$

we obtain $\lim_{v \to -\infty} K'_{min}(v) = 1$. And since both $K(v)$ and $K_{min}(v; v_0)$ are strictly concave, we must have $K'_{min}(v; v_0) \leq K'(v)$. Hence,

$$\lim_{v \to -\infty} K'_{min}(v) = 1 \leq \lim_{v \to -\infty} K'(v)$$

Therefore, by the strict concavity of $K(\cdot)$, $K_{min}(\cdot)$, and $K_{max}(\cdot)$ and since $K_{min}(v; v_0) \leq K(v) \leq K_{max}(v)$ for all $v, v_0 \in V$ we obtain our final result (ii):

$$\lim_{v \to -\infty} K'_{min}(v) = 1 \leq \lim_{v \to -\infty} K'(v) \leq 1 = \lim_{v \to -\infty} K'_{max}(v) \quad \Longrightarrow \quad \lim_{v \to -\infty} K'(v) = 1$$

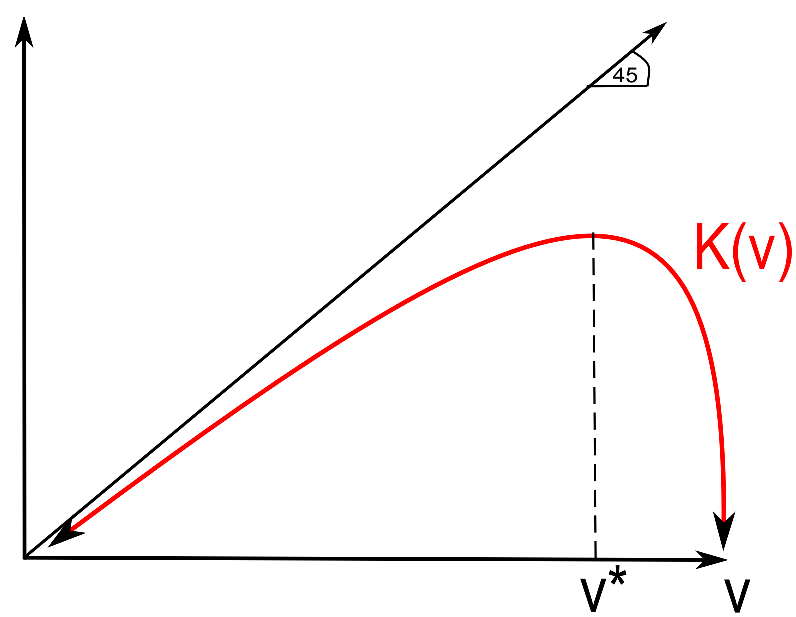
□

We can immediately make the next claim.

Claim 3.3.3
There exists a $v^* \in IntV$, where Int is the interior of the set, such that $K'(v^*) = 0$.

Proof. This follows from a straight forward application of the Intermediate Value Theorem. □

To gain better intuition for the last set of claims, the following figure graphs the Bellman.



Given the above properties of the Bellman we can take first order conditions of the program.

For all $i \in \{2, \dots, N-1\}$ the FOCs are:

$$\pi(\theta_i)\Big[\eta C'\Big(u(\theta_i|v)\Big) - (1+\lambda)\theta_i\Big] = \mu_i\theta_i - \mu_{i-1}\theta_{i-1}$$

$$\pi(\theta_i)\Big[\hat{\beta}K'\Big(w(\theta_i|v)\Big) + \beta\lambda\Big] = \beta(\mu_{i-1} - \mu_i)$$

For $i \in \{1, N\}$ the first order conditions with respect to $w(\theta|v)$ are:

$$\pi(\theta_1)\Big[\hat{\beta}K'\Big(w(\theta_1|v)\Big) + \beta\lambda\Big] = -\beta\mu_1$$

$$\pi(\theta_N)\Big[\hat{\beta}K'\Big(w(\theta_N|v)\Big) + \beta\lambda\Big] = \beta\mu_{N-1}$$

For $i \in \{1, N\}$ the first order conditions with respect to $u(\theta|v)$ are:

$$\pi(\theta_1)\Big[\eta C'\Big(u(\theta_1|v)\Big) - (1+\lambda)\theta_1\Big] = \mu_1\theta_1$$

$$\pi(\theta_N)\Big[\eta C'\Big(u(\theta_N|v)\Big) - (1+\lambda)\theta_N\Big] = -\mu_{N-1}\theta_{N-1}$$

The envelope conditions gives:

$$K'(v) = -\lambda$$

I will now use the FOCs and EC to prove a series of claims that will culminate in the no immiseration result.

Claim 3.3.4 (Mean Reversion)
Define $v_{t+1}(\theta^t) = w(\theta_t|v_t(\theta^{t-1}))$. The stochastic process for $\{K'(v_t)\}$ is mean reverting with drift toward zero.

Proof. Substituting the envelope condition into the FOC with respect to $w(\theta|v)$ and summing over θ_i we obtain:

$$\sum_{i=1}^{N} \pi(\theta_i)K'\Big(w(\theta_i|v)\Big) - \frac{\beta}{\hat{\beta}}K'(v) = -\frac{\beta}{\hat{\beta}}\sum_{i=1}^{N}(\mu_i - \mu_{i-1})$$

where $\mu_N = \mu_0 = 0$. Therefore we can rewrite the condition as:

$$\sum_{i=1}^{N} \pi(\theta_i)K'\Big(w(\theta_i|v)\Big) = \frac{\beta}{\hat{\beta}}K'(v) \tag{4}$$

From claim 3.3.3 we know that there exists $v^* \in V$ such that $K'(v^*) = 0$. So for $v > v^*$ equation (4) tells us that $K'(v) < 0$ and:

$$\sum_{i=1}^{N} \pi(\theta_i)K'\Big(w(\theta_i|v)\Big) > K'(v)$$

Furthermore, for $v < v^*$ equation (4) tells us that $K'(v) > 0$ and:

$$\sum_{i=1}^{N} \pi(\theta_i) K'\Big(w(\theta_i|v)\Big) < K'(v)$$

Because $\beta/\hat{\beta} < 1$ and since $K'(v^*) = 0$ the drift of this mean reverting process is toward zero. □

Compare the above result to the Atkeson and Lucas environment's equation (3). In their environment the stochastic process for $\{K'(v_t)\}$ followed a non-negative super martingale. Due to this property, the optimal provision of incentives required that promise utilities exhibit a spreading property: the highest promise utility for tomorrow is greater than today's and the lowest promise utility for tomorrow is less than todays. In the limit of the process $\{v_t\}$, the only way to enforce these incentives was to push *all* consumption to the future so that agents live in misery today.

In Farhi and Werning's environment, the optimal provision of incentives does not exhibit a spreading property. Instead, it exhibits mean reversion: when today's promise utility is above a threshold v^*, the planner promises less utility for tomorrow. When today's promise utility is below v^* the planner promises more utility for tomorrow. Another way to think about the provision of incentives is to recall that the planner wants agents to push consumption to the future ($\hat{\beta} > \beta$), but agents wish to consume more today. Then a natural reward arises in granting agents more consumption today when they truthfully report their type, despite the planner's wish to withhold current consumption.

In order to see the mean reversion property from another perspective, notice that by adding $(1 - \beta/\hat{\beta})$ to each side we can alternatively redefine $1 - K'(\cdot)$ as the random variable, which now has mean reverting drift toward one:

$$\sum_{i=1}^{N} \pi(\theta_i) \Big[1 - K'\Big(w(\theta_i|v)\Big)\Big] = \frac{\beta}{\hat{\beta}}\Big(1 - K'(v)\Big) + \Big(1 - \frac{\beta}{\hat{\beta}}\Big)$$

I will now prove that there are limits to providing incentives through the use of entitlements and that the no immiseration result follows from these limits.

Claim 3.3.5 (Limits to Promise Utility)
There exist boundaries on $K(v)$, such that for all $v \in V$ and $\theta \in \Theta$:

$$\underline{\gamma}\Big(1 - K'(v)\Big) + \Big(1 - \frac{\beta}{\hat{\beta}}\Big) \leq 1 - K'\Big(w(\theta|v)\Big) \leq \overline{\gamma}\Big(1 - K'(v)\Big) + \Big(1 - \frac{\beta}{\hat{\beta}}\Big)$$

where $(\underline{\gamma}, \overline{\gamma}) = \left(\frac{\beta}{\hat{\beta}} \frac{1}{\theta_{N-1}}, \frac{\beta}{\hat{\beta}} \frac{1}{\theta_1}\right)$.

Proof. We will construct the bounds from the first order conditions. Let's first derive the *upper bound*. By summing over the FOCs with respect to $u(\theta|v)$ we obtain:

$$\sum_{i=1}^{N} \pi(\theta_i) \Big[\eta C'\Big(u(\theta_i|v)\Big) - (1 - \lambda)\theta_i\Big] = \sum_{i=1}^{N} \Big[\mu_i \theta_i - \mu_{i-1}\theta_{i-1}\Big]$$

Note that $\mu_0 = \mu_N = 0$ and therefore the RHS is equal to zero. Furthermore substituting the envelope condition and since $\sum_i^N \pi(\theta_i)\theta_i = 1$:

$$\eta \sum_{i=1}^{N} \pi(\theta_i) C'\Big(u(\theta_i|v)\Big) = 1 - K'(v) \tag{5}$$

From the first order condition with respect to $u(\theta_1|v)$ and from equation (5) above we obtain:

$$1 - K'(v) = \eta \sum_{i=1}^{N} \pi(\theta_i) C'\Big(u(\theta_i|v)\Big) \geq \eta C'\Big(u(\theta_1|v)\Big) = \frac{\mu_1}{\pi(\theta_1)}\theta_1 + (1 - K - (v))\theta_1$$

$$\Big(\frac{1}{\theta_1} - 1\Big)\Big(1 - K'(v)\Big) \geq \frac{\mu_1}{\pi(\theta_1)}$$

From the FOC with respect to $w(\theta_1|v)$, substitute the above line:

$$\begin{aligned} K'\Big(w(\theta_1|v)\Big) &= \frac{\beta}{\hat{\beta}}\left[K'(v) - \frac{\mu_1}{\pi(\theta_1)}\right] \\ &\geq \frac{\beta}{\hat{\beta}}\left[K'(v) - \Big(\frac{1}{\theta_1} - 1\Big)\Big(1 - K'(v)\Big)\right] \\ &= \frac{\beta}{\hat{\beta}}\Big(1 - \frac{1}{\theta_1}\Big) + \frac{\beta}{\hat{\beta}}\frac{1}{\theta_1}K'(v) \end{aligned}$$

Multiply both sides by -1 and add 1 to both sides:

$$\begin{aligned} 1 - K'\Big(w(\theta_1|v)\Big) &\leq 1 - \frac{\beta}{\hat{\beta}}\Big(1 - \frac{1}{\theta_1}\Big) - \frac{\beta}{\hat{\beta}}\frac{1}{\theta_1}K'(v) \\ &= \Big(1 - \frac{\beta}{\hat{\beta}}\Big) + \Big(\frac{\beta}{\hat{\beta}}\frac{1}{\theta_1}\Big)\Big(1 - K'(v)\Big) \\ &\equiv \Big(1 - \frac{\beta}{\hat{\beta}}\Big) + \overline{\gamma}\Big(1 - K'(v)\Big) \end{aligned}$$

Now let's derive the *lower bound*. From the first order condition with respect to $u(\theta_N|v)$ and equation (5) we obtain:

$$1 - K'(v) = \eta \sum_{i=1}^{N} \pi(\theta_i) C'\Big(u(\theta_i|v)\Big) \leq \eta C'\Big(u(\theta_N|v)\Big) = -\frac{\mu_{N-1}}{\pi(\theta_N)}\theta_{N-1} + (1 - K - (v))\theta_{N-1}$$

$$\Big(1 - \frac{1}{\theta_{N-1}}\Big)\Big(1 - K'(v)\Big) \geq \frac{\mu_{N-1}}{\pi(\theta_N)}$$

From the FOC with respect to $w(\theta_N|v)$, substitute the above line:

$$K'\Big(w(\theta_N|v)\Big) = \frac{\beta}{\hat{\beta}}\left[K'(v) + \frac{\mu_{N-1}}{\pi(\theta_N)}\right]$$

$$\leq \frac{\beta}{\hat{\beta}} \left[K'(v) + \left(1 - \frac{1}{\theta_{N-1}}\right) \left(1 - K'(v)\right) \right]$$

$$= \frac{\beta}{\hat{\beta}} \left(1 - \frac{1}{\theta_{N-1}}\right) + \frac{\beta}{\hat{\beta}} \frac{1}{\theta_{N-1}} K'(v)$$

Multiply both sides by -1 and add 1 to both sides:

$$1 - K'\Big(w(\theta_N|v)\Big) \geq 1 - \frac{\beta}{\hat{\beta}} \left(1 - \frac{1}{\theta_{N-1}}\right) - \frac{\beta}{\hat{\beta}} \frac{1}{\theta_{N-1}} K'(v)$$

$$= \left(1 - \frac{\beta}{\hat{\beta}}\right) + \left(\frac{\beta}{\hat{\beta}} \frac{1}{\theta_{N-1}}\right) \left(1 - K'(v)\right)$$

$$\equiv \left(1 - \frac{\beta}{\hat{\beta}}\right) + \underline{\gamma}\Big(1 - K'(v)\Big)$$

Lastly, note that since $w(\theta_N|v) \leq w(\theta_{N-1}|v) \leq \cdots \leq w(\theta_1|v)$ for all $v \in V$, and since $K(\cdot)$ is strictly concave, the derived bounds are in fact bounds *for all* $\theta \in \Theta$. □

Lastly the no immiseration result.

Claim 3.3.6 (No Immiseration)
There exists an invariant distribution ψ^* with a measure zero of agents at misery: $\psi^*(\{infV\}) = 0$.

Proof. For the existence of an invariant distribution I leave the reader to look at the proof of Proposition 3 on page 397. Instead I will focus on the no immiseration result.

Notice that the difference between the boundaries in claim 3.3.5 is increasing with $v \in V$. Accordingly, the difference between the bounds disappears as $v \to -\infty$. This occurs since $\lim_{v\to-\infty} K'(v) = 1$. Thus:

$$\underline{\gamma}\Big(1 - \lim_{v\to-\infty} K'(v)\Big) + \left(1 - \frac{\beta}{\hat{\beta}}\right) \leq 1 - \lim_{v\to-\infty} K'\Big(w(\theta|v)\Big) \leq \overline{\gamma}\Big(1 - \lim_{v\to-\infty} K'(v)\Big) + \left(1 - \frac{\beta}{\hat{\beta}}\right)$$

$$\left(1 - \frac{\beta}{\hat{\beta}}\right) \leq 1 - \lim_{v\to-\infty} K'\Big(w(\theta|v)\Big) \leq \left(1 - \frac{\beta}{\hat{\beta}}\right)$$

Thus the last line implies that $w(\theta|v)$ is bounded away from $infV = -\infty$, for all $\theta \in \Theta$:

$$\lim_{v\to-\infty} K'\Big(w(\theta|v)\Big) = \frac{\beta}{\hat{\beta}} < 1 = \lim_{v\to-\infty} K'(v)$$

Therefore the invariant distribution has no mass at misery: $\psi^*(\{infV\}) = 0$. □

The key ingredient for this result is the shape of the value function, $K(v)$, which in turn depends crucially on the convexity of $C(u)$. It is precisely the value function's shape that generates mean reversion - as opposed to the monotonically increasing value function in Atkeson and Lucas, which generates a spreading property. In fact, the mean reverting process' drift toward zero is strong enough to overcome the possibility of any positive measure of agents living in misery.

3.4 Exercises: Prelim Questions

To be completed.

4 References

1. Atkeson, Andrew and Robert Lucas (1992) "On Efficient Distribution with Private Information." RES

2. Chari, V.V. and Larry Jones (2000) "A Reconsideration of Social Cost." Economic Theory

3. Diamond, Douglas and Philip Dybvig (1983) "Bank Runs, Deposit Insurance, and Liquidity ." JPE

4. Diamond, Peter and James Mirrlees (1971) "Optimal Taxation and Public Production I: Production Efficiency." AER

5. Farhi, Emmanuel and Ivan Werning (2007) "Inequality and Social Discounting." JPE

6. Golosov, Mikhail and Aleh Tsyvinski (2004) "Designing Optimal Disability Insurance: A Case for Asset Testing." JPE

7. Kocherlakota, Narayana (2003) "Zero Expected Wealth Taxes: A Mirrlees Approach to Dynamic Optimal Taxation." EMA

8. Mailath, George and Andrew Postlewaite (1990) "Asymmetric Information Bargaining Problems with Many Agents." REStud

9. Mas-Colell, Andreu; Michael Whinston and Jerry Green (1995) *Microeconomic Theory.*

10. Mirrlees, James (1971) "An Exploration in the Theory of Optimum Income Taxation." RES

11. Myerson, Roger (1979) "Incentive Compatibility and the Bargaining Problem." EMA

12. Myerson, Roger (1981) "Optimal Auction Design" Mathematics of Operations Research

13. Myerson, Roger and Mark Satterthwaite (1983) "Efficient Mechanisms for Bilateral Trading" JET

14. Rafael, Rob (1989) "Pollution Claim Settlements Under Private Information." JET

15. Stokey, Nancy; Robert Lucas and Edward Prescott (1989) *Recursive Methods in Economic Dynamics.*