Optimal Public Debt with Life Cycle Motives*

William B. Peterman
Federal Reserve Board
william.b.peterman@frb.gov

Erick Sager
Bureau of Labor Statistics
sager.erick@bls.gov

February 14, 2017

Abstract

In a seminal paper, Aiyagari and McGrattan (1998) find that it is optimal for the U.S. government to hold a large amount of public debt. Their point of reference is a standard incomplete markets model with infinitely lived agents, in which the optimality of public debt derives from how an increase in the interest rate encourages more steady state precautionary savings. This paper revisits this result in a life cycle model and finds that public debt's insurance enhancing mechanism is, now, severely limited. While public debt increases the average infinitely lived agent's *ex ante* wealth, it has much less benefit in a life cycle model since agents enter the economy with no wealth. Thus, instead of *debt* being optimal as it is in the infinitely lived agent model, when a life cycle is introduced it is optimal for the government to *save* on the order of magnitude of 59% of output. Furthermore, we find that not accounting for life cycle features when solving for optimal policy reduces welfare by nearly one-half percent of expected lifetime consumption.

Keywords: Government Debt; Life Cycle; Heterogeneous Agents; Incomplete Markets

JEL Codes: H6, E21, E6

*The authors thank Chris Carroll, William Gale, Toshi Mukoyama, Marcelo Pedroni and participants of the 2017 ASSA Meetings, Quantitative Society for Pension Studies Summer Workshop, Annual Conference of the National Tax Association, Spring 2016 Midwest Macro Meetings, and International Conference on Computing in Economics and Finance for insightful comments and discussions. The views expressed herein are those of the authors and not necessarily those of the Federal Reserve Board of Governors, the Federal Reserve System, the Bureau of Labor Statistics or the US Department of Labor.

1 Introduction

In the decades preceding the Great Recession, debt to GDP ratios in advanced economies averaged over 40%. Moreover, only three advanced economies held a net level of public savings. Motivated by these basic facts, this paper examines the optimality of public debt in the U.S. economy.

In their seminal work, Aiyagari and McGrattan (1998) find that it is optimal for the government to hold a large amount of public debt, on the order of magnitude of two-thirds the size of GDP. Their framework is the standard incomplete markets model, in which infinitely lived households can only partially insure against the realization of idiosyncratic labor productivity shocks. Using this model, Aiyagari and McGrattan (1998) show that imperfect insurance against ex post labor market outcomes admits a role for government policy to improve upon the competitive equilibrium allocation. Higher government debt (or lower government savings) tends to crowd out the stock of productive capital leading to a higher interest rate and lower wage. The relatively higher interest rate encourages households to hold more wealth, which in turn helps agents to better insure against labor earnings risk and avoid binding liquidity constraints.

This paper examines whether public debt remains optimal in a life cycle model. Given that introducing a life cycle can fundamentally alter households' savings patterns, a life cycle may change both the effectiveness and benefit of public debt in encouraging households to hold more wealth, thereby changing optimal policy. In order to determine the effect of the life cycle, we compute optimal policy in two model economies that are calibrated to be consistent with post-war U.S. macroeconomic aggregates. The first model is similar to that in Aiyagari and McGrattan (1998) and includes infinitely lived agents. The second model includes life cycle features such as a finite lifespan, mortality risk, an age-dependent wage profile, retirement and a Social Security program. We find that the optimal policies are strikingly different in the two models. In the infinitely lived agent model it is optimal for the government to hold debt equal to approximately 20 percent of output. In contrast, in the life cycle model, we find that it is no longer optimal for the government to hold debt. Instead, it is optimal for the government to hold public savings equal to almost 60 percent of output. Not only does the optimal policy look quite different when one ignores life cycle features, but the welfare consequences of ignoring them are economically significant. We find that using the life cycle model, if a government implemented the infinitely lived agent model optimal 20% debt-to-output policy, then life cycle agents would be worse off by nearly 0.5 percent of expected lifetime consumption. Overall, this paper demonstrates that incorporating life cycle features fundamentally changes whether it is optimal for the government to hold public savings or public debt.

The starkly different optimal policies can be explained, in large part, by life cycle agents' special progression through distinct phases over their life times. Specifically, life cycle model agents begin their life with no savings and enter an *accumulation phase* in which they build a precautionary stock of savings to insure against income shocks and finance their post-retirement consumption. In middle life, agents may enter a *stationary phase* in which they have accumulated a target level of assets, around which savings fluctuates.[1] Finally, older agents enter a *deaccumulation phase* in which they spend down their savings in anticipation of death. In the infinitely lived agent model, agents do not experience an accumulation phase but instead experience a perpetual stationary phase.

Using our life cycle model, we demonstrate that agents' progression through distinct lifetime phases is the underlying mechanism leading to a different optimal policy. In particular, if the government holds more public debt, then in the infinitely lived agent model the *steady state* level of aggregate savings is larger and the average agent has more wealth *ex ante*. In contrast, life cycle agents begin working life with zero wealth. Although a higher interest rate may increase the level of average savings in the stationary phase, life cycle agents' initial wealth does not respond to policy and agents immediately begin saving in the accumulation phase. Thus, ultimately, the existence of the accumulation phase is the predominant reason for the drastically different optimal policies between the two models.

[1] In life cycle models where agents live for a short enough span, agents sometimes transition directly from the accumulation phase to the deaccumulation phase skipping this stationary phase. We generally find this to be the case in our baseline life cycle model.

Our paper is related to a well established literature that has examined the optimal level of government debt and savings in quantitative models. Following Aiyagari and McGrattan (1998), a number of studies examine the optimal level of debt in the steady state of an infinitely lived agent model. Floden (2001) finds that increasing government debt can provide welfare benefits if transfers are below optimal levels. Similarly, Dyrda and Pedroni (2016) find that it is optimal for the government to hold debt. However they find that when optimizing both taxes and debt at the same time leads to a smaller level of optimal debt than previous studies. In contrast, Röhrs and Winter (2016) find that when they include a skewed wealth distribution that more closely matches the upper tail of the U.S. wealth distribution, it is optimal for the government to save as opposed to holding debt. In contrast to these papers, we study optimal public debt and savings in a life cycle model in which individuals begin life as liquidity constrained savers and grow to become retirees that run down their savings. We find that this age-dependent savings pattern can lead to different welfare effects from government savings and debt.

Our paper is also related to recent work by Dyrda and Pedroni (2016), Röhrs and Winter (2016), and Desbonnet and Weitzenblum (2012), that finds quantitatively large welfare costs of transitioning between steady states after a change in public debt. However, there are two differences between this paper and the other papers. First, these studies focus on models inhabited by infinitely lived agent and do not incorporate the mechanisms prevalent in a life cycle setting. Second, we do not consider the transitions, and instead focuses on steady state comparisons to more sharply highlight the contribution of the life cycle to the question of optimal debt and welfare.

Lastly, our paper is related to Dávila, Hong, Krusell, and Ríos-Rull (2012), whose work defines constrained inefficiency in a standard incomplete markets model and shows what forces lead to the optimality of a larger aggregate capital stock. Our paper examines a different model and achieves improved welfare through the restriction of government policy to debt policy. Despite these differences, our paper arrives at a similar conclusion that the current U.S. capital stock is too low.

The remainder of this paper is organized as follows. Section 2 illustrates the underlying mechanisms by which optimal government policy interacts with life cycle and infinitely lived agent model features. Section 3 describes the life cycle and infinitely lived agent model environments and defines equilibrium. Section 4 presents the calibration strategy and Section 5 presents quantitative results. Section 6 concludes.

2 Illustration of the Mechanisms

In this section, we discuss the mechanisms that might lead the government to hold debt or savings. We begin by discussing the core inefficiency that provides a welfare improving role for public savings or public debt. Then we turn to why the optimal policy may differ in the two models. Specifically, we discuss how life cycle features generate a pattern of savings over the life cycle that is distinct from average savings in the infinitely lived agent model. Finally, we discuss the main channels by which government debt or savings impacts individual behavior and how the strength of these channels may vary between the life cycle and infinitely lived agent models.

2.1 Pecuniary Externality

Individual agents, who are constrained by incomplete asset markets and borrowing constraints, do not internalize how their decisions impact prices. In such an environment, the price mechanism does not fully work and competitive equilibria are generically inefficient. If agents were to systematically deviate from individual optimization, then equilibrium prices could be attained that improve social welfare.

The government's public savings policy can attain higher welfare than the competitive equilibrium allocation by partially correcting this pecuniary externality. By choosing a public savings (debt) policy, the government *crowds in (out)* the supply of loanable funds that can be directed to firms for investment in productive capital. The government's public savings (debt) policy directly changes the supply (demand) for assets and therefore manipulates the equilibrium interest rate through market clearing conditions. Because it understands the relationship between public savings and factor prices, the government can implement a welfare improving allocation that individual agents could not attain alone.

To illustrate the nature of this externality, we derive the government's optimality condition for public savings. For ease of explication, consider a simplified model in which agents live for J periods with certainty, value consumption and hours according to standard utility functions $u(c)$ and $v(h)$, respectively, and discount the future with $\beta < 1$. Each period, agents consume, save and work while collecting labor and asset income at given prices w and r, respectively. Labor productivity, denoted e, is subject to random shocks, denoted by ε. Lastly, suppose that the government does not spend $(G = 0)$ and can choose any level of public savings B'.

The government's problem is to maximize the present value of expected utility of an agent prior to entering the economy, subject to allocations being a competitive equilibrium. The government understands how individual allocations and prices vary with public savings.

$$S(B) \equiv \max_{B' \in \mathbb{R}} \sum_{\varepsilon} \int_A \mathbb{E}_1 \sum_{j=1}^{J} \beta^{j-1} \left[u\left(c_j(a,\varepsilon;B')\right) - v\left(h_j(a,\varepsilon;B')\right) \right] d\lambda_1(a,\varepsilon)$$

$$\begin{aligned} \text{s.t.} \quad c_j(a,\varepsilon;B) + a'_j(a,\varepsilon;B') &= we(\varepsilon)h_j(a,\varepsilon;B) + (1+r)a \\ w &= F_L(K,L) \\ r &= F_K(K,L) - \delta \\ K &= \textstyle\sum_{j=1}^{J} \sum_{\varepsilon} \int_A a d\lambda_j(a,\varepsilon;B) + B \\ L &= \textstyle\sum_{j=1}^{J} \sum_{\varepsilon} \int_A e(\varepsilon)h_j(a,\varepsilon;B) d\lambda_j(a,\varepsilon;B) \end{aligned}$$

where $\lambda_1(a,\varepsilon)$ is the joint distribution over wealth and labor productivity shocks, and (K, L) are aggregate capital and labor. Therefore, the government's optimality condition is:[2]

$$\sum_{j=1}^{J} \mu_j \sum_{\varepsilon} \int_{\mathcal{A}} \omega_j \beta \sum_{\varepsilon'} \pi_j(\varepsilon'|\varepsilon) u'(c_{j+1}) \left(\frac{a'_j}{K'} - \frac{e(\varepsilon')h_{j+1}}{L'} \right) d\lambda_j(a,\varepsilon) = 0$$

[2]Derivations are provided in Appendix A.1.

where ω_j is given by:

$$\omega_j \equiv \frac{dr'}{dK'} \cdot \frac{dK'_j}{dB'} \cdot K' - \frac{dw'}{dL'} \cdot \frac{dL'_j}{dB'} \cdot L'$$

This ω_j term captures the effect of public savings on prices by way of changes in aggregate factor inputs. A change in public savings dB' induces a change in factor prices given by $dr/dK' = F_{KK}(K', L')$ and $dw/dL' = F_{LL}(K', L')$, and a direct change in aggregate capital and aggregate labor given by:

$$\frac{dK'_j}{dB'} = \frac{d}{dB'}(A'_j + B') = \sum_{\varepsilon'} \sum_{\varepsilon} \pi_j(\varepsilon'|\varepsilon) \int \frac{da'_j(a,\varepsilon)}{dB'} d\lambda_j(a,\varepsilon) + 1$$

$$\frac{dL'_j}{dB'} = \sum_{\varepsilon'} \int e(\varepsilon') \frac{dh_{j+1}(a',\varepsilon')}{dB'} d\lambda_{j+1}(a',\varepsilon').$$

If the competitive equilibrium were efficient, then $\omega_j = 0$ and a change in the government's public savings policy would be socially suboptimal. However, $\omega_j = 0$ does not generically hold in competitive equilibrium.

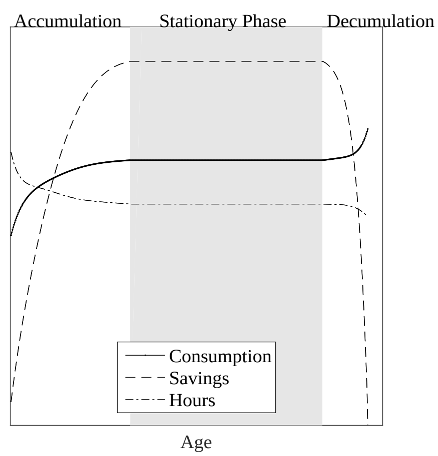
Instead, the government chooses policy according to a marginal utility weighted average of the individual effects of an increase in public savings. This means that the government places higher weight on low income agents who have high marginal utility of consumption. For example, low wealth agents may choose to work more hours despite low labor productivity, in which case the government assigns a high marginal utility weight to a $a'_j/K' < e_{j+1}h_{j+1}/L'$. The aggregate effect depends on the distribution $\lambda_j(a, \varepsilon)$.

While the simplified model illustrates the tradeoffs that determine optimal public debt, we cannot provide a tighter characterization of optimal policy without imposing additional assumptions on the problem. Instead of pursuing a characterization of the simplified model further, we will examine the distinct features in the life cycle model that can lead to a different optimal policy than that of the infinitely lived agent model. We will focus on how allocations vary over a life cycle in Section 2.2, and what life cycle features imply for the effect of public debt on welfare in Section 2.3.

2.2 Life Cycle Phases

In order to highlight how the life cycle may impact optimal debt policy, it will be useful to consider the following illustrative example. Suppose that agents are born with zero wealth, work throughout their lifetimes and die with certainty within a finite number of periods. Agents face idiosyncratic labor productivity shocks and use assets to partially insure against the resulting earnings risk. Since agents do not retire, savings is accumulated only to insure against idiosyncratic shocks.

Figure 1: Illustrative example of life cycle phases.



For this hypothetical economy, Figure 1 depicts cross-sectional averages for savings, hours and consumption decisions at each age. Figure 1 shows that agents experience three different phases. Agents enter the economy without any wealth and begin the *accumulation phase*, which is characterized by the accumulation of wealth for precautionary motives. While accumulating a stock of savings, agents tend to work more and consume less.

Once a cohort's average wealth provides sufficient insurance against labor productivity shocks, these agents have entered the *stationary phase*.[3] This

[3] The stationary level of average savings is related to the "target savings level" in Carroll (1992, 1997). Given the primitives of the economy, an agent faces a tradeoff between consumption levels and consumption smoothing. The agent targets a level of savings that provides sufficient insurance while maximizing expected consumption.

phase is characterized by savings, hours and consumption that remain constant in the aggregate. However, underlying constant aggregates are agents who respond to shocks by choosing different allocations and moving about various states within a non-degenerate distribution of savings, hours and consumption.

Finally, agents enter the *deaccumulation phase* as they approach the end of their lives. In order to smooth consumption in the final periods of their lives, agents attempt to deaccumulate assets so that they are not forced to consume a large quantity immediately preceding death. Furthermore, with few periods of life remaining, agents no longer want to hold as much wealth for precautionary reasons. Thus, the average level of savings and labor supply decreases, while consumption increases slightly.

2.3 Channels By Which Public Debt Affects Agents

In this section we identify three main channels through which public debt policy affects welfare: the *level channel*, the *insurance channel*, and the *inequality channel*. The level channel is a direct effect of policy on aggregate resources. This channel can be thought of as affecting welfare by changing the welfare for the average agent. The remaining channels operate indirectly through general equilibrium effects such as market clearing prices. In contrast to the level channel, the additional two channels affect welfare by changing the distribution of resources. We heuristically characterize how these channels differ across life cycle and infinitely lived agent economies and lead to different optimal policies.

Level Channel: Generally, shifting from public debt to public savings creates more productive capital and, through a higher wage, encourages agents to work more hours. The higher level of both aggregate productive capital and labor will tend to lead to higher consumption.[4] The increase in aggregate

[4]While public savings can crowd out private savings, our quantitative findings show that the elasticity of private savings to public savings is less than one. Therefore we find that public savings increases productive capital across models. Additionally, the production technology is increasing in aggregate capital. Because we assume that the production technology exhibits decreasing marginal returns to capital, aggregate consumption and aggregate private

consumption will tend to increase welfare while the increase in aggregate labor will tend to decrease welfare. Thus, the overall effect of the levels channel will depend on the relative magnitude of these competing effects. Furthermore, there is no a priori argument for why these relative size of these effects should differ in the life cycle or infinitely lived agent economies.

Insurance Channel: If the government holds more public debt this mechanically crowds out productive capital and leads to a higher interest rate in asset markets. The higher interest rate will tend to be accompanied by agents holding a larger level of precautionary savings. All else equal, the larger level of precautionary savings can lead to an increase in welfare because agents are better insured against labor earnings risk since they are less likely to face binding liquidity constraints.

The link between public debt and the benefit of the insurance channel is fundamentally different in the infinitely lived and life cycle model. In the infinitely lived agent model, agents live in a perpetual stationary phase. Thus, if the government holds more public debt the *steady state* level of aggregate savings will be larger implying that the average agent has more wealth *ex ante*. In contrast, in the life cycle model agents enter the model with zero wealth and immediately begin the accumulation phase.[5] Thus, if the government holds more public debt then the average agent may hold more savings over their lifetime however there will be the cost of accumulating that savings over their lifetime. Overall, without the accumulation cost, the benefit from the insurance channel will tend to be larger in the infinitely lived agent model.

Inequality Channel: From an ex ante perspective, ex post income inequality lowers utility since agents are risk averse. This income inequality is generated by inequality in both interest and labor income. Since changing public

savings may decrease when the aggregate capital stock is sufficiently large. In our quantitative experiments, however, we find that aggregate consumption is increasing in public savings when public savings varies between the calibrated debt level and the optimal policy.

[5] If life cycle features were introduced in a dynastic model, instead of a life cycle model, where old agents bequeath wealth to agents entering the economy, then the accumulation phase may be more responsive to public policy. Consistent with Fuster, Imrohoroglu, and Imrohoroglu (2008), the optimal policy differences with the infinitely lived agent model could be smaller since agents would receive some initial wealth through bequests.

debt has the opposite effects on the factor prices from the two sources of income, debt policy can be used to reduce the spread in lifetime income across agents. For example, if labor income is a larger contributor to inequality in lifetime income then increasing public debt and lowering the wage will tend to decrease overall lifetime income inequality.[6]

As demonstrated in Dávila, Hong, Krusell, and Ríos-Rull (2012), the contribution to overall inequality from labor versus interest income can depend on the length of an agent's lifespan. In the extreme, in a model where agents live for one period labor earnings constitute all of total income and will be the total source of income inequality. In this case, the government would want to enact a debt policy that lowers the wage rate in order to reduce total income inequality (more public debt). As the length of the lifetime extends, interest income becomes a contributor to overall income inequality. In particular, as the length of the lifetime increases there is an increase in overall total lifetime labor income inequality due to a positive probability that agents receive a long string of either positive or negative productivity shocks. As a result more asset income inequality will also develop because agents reduce their wealth in response to a string of negative shocks and increase their wealth in response to a string of positive shocks. Thus, if as the length of the lifetime increases the relative contribution to lifetime income inequality from interest income increases the government may want to move to a debt policy that leads to a relatively lower interest rate (more public savings). Thus, the inequality channel may lead the government towards holding public debt in life cycle model and public savings in an infinitely lived agent model.

3 Economic Environment

In this section, we present both the Life Cycle model and the Infinitely Lived Agent model. Given that there are many common features across models, we will first focus on the Life Cycle model in detail before providing an overview of the Infinitely Lived Agent model.

[6]This change in public debt will increase the interest rate and introduce more inequality from interest income. However, this effect will tend to be dominated by the lower of labor income inequality.

3.1 Life Cycle Model

3.1.1 Production

Assume there exist a large number of firms that sells goods in perfectly competitive product markets, purchase inputs from perfectly competitive factor markets and each operate an identical constant returns to scale production technology, $Y = ZF(K, L)$. These assumptions on primitives admit a representative firm. The representative firm chooses capital (K) and labor (L) inputs in order to maximize profits, given an interest rate r, a wage rate w, a level of total factor productivity Z and capital depreciation rate $\delta \in (0, 1)$.

3.1.2 Consumers

Demographics: Let time be discrete and let each model period represent a year. Each period, the economy is inhabited by J overlapping generations of individuals. Age J is each agent's exogenous terminal period of life. Before period J all living agents face mortality risk. Conditional on living to age j, agents have a probability s_j of living to age $j+1$, with a terminal age probability given by $s_J = 0$. Each period a new cohort is born and the size of each successive newly born cohort grows at a constant rate $g_n > 0$.

Agents who die before age J may hold savings since mortality is uncertain. These savings are treated as accidental bequests and are equally divided across each living agent in the form of a lump-sum transfer, denoted Tr.

Preferences: Agents rank lifetime paths of consumption and labor, denoted $\{c_j, h_j\}_{j=1}^{J}$, according to the following preferences:

$$\mathbb{E}_1 \sum_{j=1}^{J} \beta^{j-1} s_j \left[u(c_j) - \zeta_j' v(h_j) \right]$$

where β is the time discount factor. Expectations are taken with respect to the stochastic processes governing labor productivity. Furthermore, $u(c)$ and $v(h)$ are instantaneous utility functions over consumption and labor hours, respectively, satisfying standard conditions. Lastly, ζ_j' is a retirement decision that is described immediately below.

Retirement: Agents choose their retirement age, which is denoted by J_{ret}. A retired agent may not sell labor hours and the decision is irreversible. Agents endogenously determine retirement age in the interval $j \in [\underline{J}_{ret}, \bar{J}_{ret}]$ and are forced to retire after age $\bar{J}_{ret}$. Let $\zeta_j' \equiv \mathbb{1}(j < J_{ret})$ denote an indicator variable that equals one when an agent chooses to continue working and zero upon retirement.

Labor Earnings: Agents are endowed with one unit of time per period, which they split between leisure and market labor. During each period of working life, an agent's labor earnings are we_jh_j, where w is the wage rate per efficiency unit of labor, e_j is the agent's idiosyncratic labor productivity drawn at age j and h_j is the time the agent chooses to work at age j.

Following Kaplan (2012), we assume that labor productivity shocks can be decomposed into four sources:

$$\log(e_j) = \kappa + \theta_j + \nu_j + \epsilon_j$$

where (i) $\kappa \overset{iid}{\sim} \mathcal{N}(0, \sigma_\kappa^2)$ is an individual-specific fixed effect that is drawn at birth, (ii) $\{\theta_j\}_{j=1}^{J}$ is an age-specific fixed effect, (iii) ν_j is a persistent shock that follows an autoregressive process given by $\nu_{j+1} = \rho\nu_j + \eta_{j+1}$ with $\eta \overset{iid}{\sim} \mathcal{N}(0, \sigma_\nu^2)$ and $\eta_1 = 0$, and (iv) $\epsilon_j \overset{iid}{\sim} \mathcal{N}(0, \sigma_\epsilon^2)$ is a per-period transitory shock.

For notational compactness, we denote the relevant state as a vector $\varepsilon_j = (\kappa, \theta_j, \nu_j, \epsilon_j)$ that contains each element necessary for computing contemporaneous labor earnings and forming expectations about future labor earnings. Denote the Markov process governing the process for ε by $\pi_j(\varepsilon_{j+1}|\varepsilon_j)$ for each $j = 1, \ldots, \bar{J}_{ret}$ and for each ε_j, ε_{j+1}. Lastly, define the function $e(\varepsilon_j) \equiv e_j$.

Insurance: Agents have access to a single asset, a non-contingent one-period bond denoted a_j with a market determined rate of return of r. Agents may take on a net debt position, in which case they are subject to a borrowing constraint that requires their debt position be bounded below by $\underline{a} \in \mathbb{R}$. Agents are endowed with zero initial wealth, such that $a_1 = 0$ for each agent.

3.1.3 Government Policy

The government (i) consumes an exogenous amount G, (ii) collects linear Social Security taxes τ_{ss} on all pre-tax labor income below an amount $\bar{x}$, (iii) distributes lump-sum Social Security payments b_{ss} to retired agents, (iv) distributes accidental bequests as lump-sum transfers Tr, and (v) taxes each individual's taxable income according to an increasing and concave function $\Upsilon(\cdot)$.

Social Security: The model's Social Security system consists of taxes and payments. The social security payroll tax is given by τ_{ss} with a per-period cap denoted by $\bar{x}$. We assume that half of the social security contributions are paid by the employee and half by the employer. Therefore, the consumer pays a payroll tax given by: $(1/2)\ \tau_{ss} \min\{weh, \bar{x}\}$. Social security payments are computed using an averaged indexed monthly earnings (AIME) that summarizes an agents lifetime labor earnings. The AIME is denoted by $\{x_j\}_{j=1}^{J}$ and is given by:

$$x_{j+1} = \left\{ \begin{array}{ll} \frac{1}{j}\left(\min\{we_j h_j, \bar{x}\} + (j-1)x_j\right) & \text{for } j \leq 35 \\ \max\left\{x_j, \frac{1}{j}\left(\min\{we_j h_j, \bar{x}\} + (j-1)x_j\right)\right\} & \text{for } j \in (35, J_{ret}) \\ x_j & \text{for } j \geq J_{ret} \end{array} \right\}$$

The AIME is a state variable for determining future benefits. Benefits consists of a base payment and an adjusted final payment. The base payment, denoted by $b^{ss}_{base}(x_{J_{ret}})$, is computed as a piecewise-linear function over the individual's average labor earnings at retirement $x_{J_{ret}}$:

$$b^{ss}_{base}(x_{J_{ret}}) = \left\{ \begin{array}{lll} \tau_{r1} & \text{for} & x_{J_{ret}} \in [0, b^{ss}_1) \\ \tau_{r2} & \text{for} & x_{J_{ret}} \in [b^{ss}_1, b^{ss}_2) \\ \tau_{r3} & \text{for} & x_{J_{ret}} \in [b^{ss}_2, b^{ss}_3) \end{array} \right\}$$

Lastly, the final payment requires an adjustment that penalizes early retirement and credits delayed retirement. The adjustment is given by:

$$b_{ss}(x_{J_{ret}}) = \left\{ \begin{array}{lll} (1 - D_1(J_{nra} - J_{ret}))b^{ss}_{base}(x_{J_{ret}}) & \text{for} & \underline{J}_{ret} \leq J_{ret} < J_{nra} - 1 \\ (1 + D_2(J_{ret} - J_{nra}))b^{ss}_{base}(x_{J_{ret}}) & \text{for} & J_{nra} \leq J_{ret} \leq \bar{J}_{ret} \end{array} \right\}$$

where $D_i(\cdot)$ are functions governing the benefits penalty or credit, $\underline{J}_{ret}$ is the earliest age agents can retire, J_{nra} is the "normal retirement age" and $\bar{J}_{ret}$ is the latest age an agent can retire.

Net Government Transfers: Taxable income is defined as labor income and capital income net of social security contributions from an employer. Denote taxable income by:

$$y(h, a, \varepsilon, \zeta) \equiv \zeta we(\varepsilon)h + r(a + Tr) - \zeta \frac{\tau_{ss}}{2} \min\{we(\varepsilon)h, \bar{x}\}$$

The government taxes each individual's taxable income according to the function $\Upsilon(y(h, a, e, \zeta))$.

Define the function $T(\cdot)$ as the government's net transfers of income taxes, social security payments and social security payroll taxes to working age agents (if $\zeta = 1$) and retired agents (if $\zeta = 0$). Net transfers are given by:

$$T(h, a, \varepsilon, x, \zeta) = (1 - \zeta)b_{ss}(x) - \zeta \frac{\tau_{ss}}{2} \min\{we(\varepsilon)h, \bar{x}\} - \Upsilon(y(h, a, \varepsilon, \zeta))$$

Public Savings and Budget Balance: Each period, the government accumulates savings, denoted B', and collects asset income rB. The resulting government budget constraint is:

$$G + B' - B = rB + \Upsilon_y$$

where Υ_y is aggregate revenues from income taxation and G is an unproductive level of government expenditures.[7] The model's Social Security system

[7] Two recent papers, Röhrs and Winter (2016) and Chaterjee, Gibson, and Rioja (2016) have relaxed the standard Ramsey assumption that government expenditures are unproductive. Both papers show that public savings is optimal with productive government expenditures, intuitively because there is an additional benefit to aggregate output.

is self-financing and therefore does not appear in the governmental budget constraint.

3.1.4 Consumer's Problem

The agent's state variables consist of asset holdings a, labor productivity shocks $\varepsilon \equiv (\kappa, \theta, \nu, \epsilon)$, Social Security contribution (AIME) variable x and retirement status ζ. For age $j \in \{1, \ldots, J\}$, the agent's recursive problem is:

$$V_j(a, \varepsilon, x, \zeta) = \max_{c, a', h, \zeta'} \left[u(c) - \zeta' v(h)\right] + \beta s_j \sum_{\varepsilon'} \pi_j(\varepsilon'|\varepsilon) V_{j+1}(a', \varepsilon', x', \zeta')$$

$$\begin{aligned} \text{s.t.} \quad c + a' &\leq \zeta' w e(\varepsilon) h + (1+r)(a + Tr) + T(h, a, \varepsilon, x, \zeta) \qquad (1) \\ a' &\geq \underline{a} \\ \zeta' &\in \{\mathbb{1}(j < \underline{J}_{ret}),\ \mathbb{1}(j \leq \bar{J}_{ret}) \cdot \zeta\} \end{aligned}$$

The indicator function $\mathbb{1}(j < \underline{J}_{ret})$ equals one when an agent is too young to retire and equals zero thereafter. Additionally $\mathbb{1}(j \leq \bar{J}_{ret})$ equals zero for all ages after an agent must retire and equals one beforehand. Therefore the agent's recursive problem nests all three stages of life: working life, near-retirement and retirement.[8]

3.1.5 Recursive Competitive Equilibrium

Agents are heterogeneous with respect to their age $j \in \mathbf{J} \equiv \{1, \ldots, J\}$, wealth $a \in \mathbf{A}$, labor productivity $\varepsilon \in \mathbf{E}$, average lifetime earnings $x \in \mathbf{X}$ and retirement status $\zeta \in \mathbf{R} \equiv \{0, 1\}$. Let $\mathbf{S} \equiv \mathbf{A} \times \mathbf{E} \times \mathbf{X} \times \mathbf{R}$ be the state space and $\mathcal{B}(\mathbf{S})$ be the Borel σ-algebra on $\mathbf{S}$. Let $\mathbf{M}$ be the set of probability measures on $(\mathbf{S}, \mathcal{B}(\mathbf{S}))$. Then $(\mathbf{S}, \mathcal{B}(\mathbf{S}), \lambda_j)$ is a probability space in which $\lambda_j(S) \in \mathbf{M}$ is a probability measure defined on subsets of the state space, $S \in \mathcal{B}(\mathbf{S})$, that describes the distribution of individual states across age-j agents. Denote the

[8] During an agent's working life (ages $j < \underline{J}_{ret}$) the agent's choice set for retirement is $\zeta' \in \{1, 1\}$ and therefore the agent must continue working. Near retirement (ages $\underline{J}_{ret} \leq j \leq \bar{J}_{ret}$), the agent's choice set is $\zeta' \in \{0, 1\}$ and the agent may retire by choosing $\zeta' = 0$. Lastly, if an agent has retired either because he chose retirement at a previous date ($\zeta = 0$) or because of mandatory retirement ($j > \bar{J}_{ret}$), then the choice set is $\{0, 0\}$ and $\zeta' = \zeta = 0$.

fraction of the population that is age $j \in \mathbf{J}$ by μ_j. For each set $S \in \mathcal{B}(\mathbf{S})$, $\mu_j \lambda_j(S)$ is the fraction of age $j \in \mathbf{J}$ and type $S \in \mathbf{S}$ agents in the economy. We can now define a recursive competitive equilibrium of the economy.

Definition (Equilibrium): Given a government policy $(G, B, B', \Upsilon, \tau_{ss}, b_{ss})$, a *stationary recursive competitive equilibrium* is (i) an allocation for consumers described by policy functions $\{c_j, a'_j, h_j, \zeta'_j\}_{j=1}^J$ and consumer value function $\{V_j\}_{j=1}^J$, (ii) an allocation for the representative firm (K, L), (iii) prices (w, r), (iv) accidental bequests Tr, and (v) distributions over agents' state vector at each age $\{\lambda_j\}_{j=1}^J$ that satisfy:

(1) Given prices, policies and accidental bequests, $V_j(a, \varepsilon, x)$ solves the Bellman equation (1) with associated policy functions $c_j(a, \varepsilon, x, \zeta)$, $a'_j(a, \varepsilon, x, \zeta)$, $h_j(a, \varepsilon, x, \zeta)$ and $\zeta'_j(a, \varepsilon, x, \zeta)$.

(2) Given prices (w, r), the representative firm's allocation minimizes cost: $r = ZF_K(K, L) - \delta$ and $w = ZF_L(K, L)$

(3) Accidental bequests, Tr, from agents who die at the end of this period are distributed equally across next period's living agents:

$$(1 + g_n)Tr = \sum_{j=1}^{J} (1 - s_j)\mu_j \int a'_j(a, \varepsilon, x, \zeta) d\lambda_j(a, \varepsilon, x, \zeta)$$

(4) Government policies satisfy budget balance:

$$G + (B' - B) = rB + \Upsilon_y \tag{2}$$

aggregate income tax revenue is given by:

$$\Upsilon_y \equiv \sum_{j=1}^{J} \mu_j \int \Upsilon \left(y\big(h_j(a, \varepsilon, x, \zeta), a, \varepsilon, \zeta'_j(a, \varepsilon, x, \zeta)\big) \right) \, d\lambda_j(a, \varepsilon, x, \zeta)$$

(5) Social security is self-financing:

$$\sum_{j=1}^{J} \mu_j \int \zeta'_j(a, \varepsilon, x, \zeta) \tau_{ss} \min\{we(\varepsilon)h_j(a, \varepsilon, x, \zeta), \bar{x}\} d\lambda_j(a, \varepsilon, x, \zeta)$$

$$= \sum_{j=1}^{J} \mu_j \int (1 - \zeta_j'(a, \varepsilon, x, \zeta)) b_{ss}(x) d\lambda_j(a, \varepsilon, x, \zeta) \tag{3}$$

(6) Given policies and allocations, prices clear asset and labor markets:

$$K - B = \sum_{j=1}^{J} \mu_j \int a \; d\lambda_j(a, \varepsilon, x, \zeta)$$

$$L = \sum_{j=1}^{J} \mu_j \int \zeta_j'(a, \varepsilon, x, \zeta) e(\varepsilon) h_j(a, \varepsilon, x, \zeta) \; d\lambda_j(a, \varepsilon, x, \zeta)$$

and the allocation satisfies the resource constraint (guaranteed by Walras' Law):

$$\sum_{j=1}^{J} \mu_j \int c_j(a, \varepsilon, x, \zeta) d\lambda_j(a, \varepsilon, x, \zeta) + G + K' = ZF(K, L) + (1 - \delta)K$$

(7) Given consumer policy functions, distributions across age j agents $\{\lambda_j\}_{j=1}^{J}$ are given recursively from the law of motion $T_j^* : \mathbf{M} \to \mathbf{M}$ for all $j \in \mathbf{J}$ such that T_j^* is given by:

$$\lambda_{j+1}(\mathcal{A} \times \mathcal{E} \times \mathcal{X} \times \mathcal{R}) = \sum_{\zeta \in \{0,1\}} \int_{A \times E \times X} Q_j \left((a, \varepsilon, x, \zeta), \mathcal{A} \times \mathcal{E} \times \mathcal{X} \times \mathcal{R} \right) d\lambda_j$$

where $\mathcal{S} \equiv \mathcal{A} \times \mathcal{E} \times \mathcal{X} \times \mathcal{R} \subset \mathbf{S}$, and $Q_j : \mathbf{S} \times \mathcal{B}(\mathbf{S}) \to [0, 1]$ is a transition function on $(\mathbf{S}, \mathcal{B}(\mathbf{S}))$ that gives the probability that an age-j agent with current state $\mathbf{s} \equiv (a, \varepsilon, x, \zeta)$ transits to the set $\mathcal{S} \subset \mathbf{S}$ at age $j + 1$. The transition function is given by:

$$Q_j \left((a, \varepsilon, x, \zeta), \mathcal{S} \right) = \left\{ \begin{array}{ll} s_j \cdot \pi_j(\mathcal{E}|\varepsilon)^{\zeta} & \text{if } a_j'(\mathbf{s}) \in \mathcal{A}, \; x_j'(\mathbf{s}) \in \mathcal{X}, \; \zeta_j'(\mathbf{s}) \in \mathcal{R} \\ 0 & \text{otherwise} \end{array} \right\}$$

where agents that continue working and transition to set $\mathcal{E}$ choose $\zeta_j'(\mathbf{s}) = 1$, while agents that transition from working life to retirement choose $\zeta_j'(\mathbf{s}) = 0$. For $j = 1$, the distribution λ_j reflects the invariant distribution $\pi_{ss}(\varepsilon)$ of initial labor productivity over $\varepsilon = (\kappa, \theta_1, 0, \epsilon_1)$.

(8) Aggregate capital, governmental debt, prices and the distribution over consumers are stationary, such that $K' = K$, $B' = B$, $w' = w$, $r' = r$, and $\lambda'_j = \lambda_j$ for all $j \in \mathbf{J}$.

3.2 Infinitely Lived Agent Model

The infinitely lived agent model differs from the life cycle model in three ways. First, agents in the infinitely lived agent model have no mortality risk ($s_j = 1$ for all $j \geq 1$) and lifetimes are infinite ($J \to \infty$). Second, labor productivity no longer has an age-dependent component ($\theta_j = 0$ for all $j \geq 1$). Lastly, there is no retirement ($\underline{J}_{ret} \to \infty$ such that $\zeta_j = 1$ for all $j \geq 1$) and there is no Social Security program ($\tau_{ss} = 0$ and $b_{ss}(x) = 0$ for all x).

Accordingly, we study a stationary recursive competitive equilibrium in which the initial endowment of wealth and labor productivity shocks no longer affects individual decisions and the distribution over wealth and labor productivity is time invariant.

Definition (Equilibrium): Given a government policy (G, B, B', Υ), a *stationary recursive competitive equilibrium* is (i) an allocation for consumers described by policy functions (c, a', h) and consumer value function V, (ii) an allocation for the representative firm (K, L), (iii) prices (w, r), and (v) a distribution over agents' state vector λ that satisfy:

(1) Given prices and policies, $V(a, \varepsilon)$ solves the following Bellman equation:

$$V(a,\varepsilon) = \max_{c,a',h} \; \big[u(c) - v(h)\big] + \beta \sum_{\varepsilon'} \pi(\varepsilon'|\varepsilon) V(a', \varepsilon')$$

$$\text{s.t.} \quad \begin{aligned} c + a' &\leq we(\varepsilon)h + (1+r)a + \Upsilon(y(h,a,\varepsilon)) \\ a' &\geq \underline{a} \end{aligned} \tag{4}$$

with associated policy functions $c(a, \varepsilon)$, $a'(a, \varepsilon)$ and $h(a, \varepsilon)$.

(2) Given prices (w, r), the representative firm's allocation minimizes cost.

(3) Government policies satisfy budget balance:

$$G + (B' - B) = rB + \Upsilon_y$$

aggregate income tax revenue is given by:

$$\Upsilon_y \equiv \int \Upsilon\big(y(h(a,\varepsilon),a,\varepsilon)\big)\ d\lambda(a,\varepsilon)$$

(4) Given policies and allocations, prices clear asset and labor markets:

$$K - B = \int a\ d\lambda(a,\varepsilon)$$

$$L = \int e(\varepsilon)h(a,\varepsilon)\ d\lambda(a,\varepsilon)$$

and the allocation satisfies the resource constraint (guaranteed by Walras' Law):

$$\int c(a,\varepsilon)d\lambda(a,\varepsilon) + G + K' = ZF(K,L) + (1-\delta)K$$

(5) Given consumer policy functions, the distribution over wealth and productivity shocks is given recursively from the law of motion $T^* : \mathbf{M} \to \mathbf{M}$ such that T^* is given by:

$$\lambda'(\mathcal{A} \times \mathcal{E}) = \int_{A \times E} Q_j\left((a,\varepsilon), \mathcal{A} \times \mathcal{E}\right) d\lambda$$

where $\mathcal{S} \equiv \mathcal{A} \times \mathcal{E} \subset \mathbf{S}$, and $Q : \mathbf{S} \times \mathcal{B}(\mathbf{S}) \to [0,1]$ is a transition function on $(\mathbf{S}, \mathcal{B}(\mathbf{S}))$ that gives the probability that an agent with current state $\mathbf{s} \equiv (a,\varepsilon)$ transits to the set $\mathcal{S} \subset \mathbf{S}$ in the next period. The transition function is given by:

$$Q\left((a,\varepsilon),\mathcal{S}\right) = \left\{ \begin{array}{cl} \pi(\mathcal{E}|\varepsilon) & \text{if } a'(\mathbf{s}) \in \mathcal{A}, \\ 0 & \text{otherwise} \end{array} \right\}$$

(6) Aggregate capital, governmental debt, prices and the distribution over consumers are stationary, such that $K' = K$, $B' = B$, $w' = w$, $r' = r$, and

$\lambda' = \lambda$.

3.3 Balanced Growth Path

Following Aiyagari and McGrattan (1998), we will further assume that total factor productivity, Z, grows over time at rate $g_z > 0$. In both the life cycle model and infinitely lived agent model, we will study a balanced growth path equilibrium in which all aggregate variables grow at the same rate as output. Denote the growth rate of output as g_y. Refer to Appendix A.2 for a formal construction of the balanced growth path for this set of economies.

4 Calibration

One subset of parameters are assigned values without needing to solve the model. The other subset of parameters are estimated using a simulated method of moments procedure that minimizes the distance between model generated moments and empirical ones. Table 1 summarizes the target and value for each parameter.

Demographics: We set the conditional survival probabilities $\{s_j\}_{j=1}^{J}$ according to Bell and Miller (2002) and impose $s_J = 0$. We set the population growth rate to $g_n = 0.011$ to match annual population growth in the US.

Production: The production function is assumed to be Cobb-Douglas of the form $F(K, L) = K^{\alpha} L^{1-\alpha}$ where $\alpha = 0.36$ is the income share accruing to capital. The depreciation rate is to $\delta = 0.0833$ which allows the model to match the empirically observed investment-to-output ratio.

Preferences: The utility function is is separable in the utility over consumption and disutility over labor:

$$u(c) - \zeta v(h) = \frac{c^{1-\sigma}}{1-\sigma} - \zeta \left(\chi_1 \frac{h^{1+\frac{1}{\gamma}}}{1+\frac{1}{\gamma}} + \chi_2 \right) .$$

Utility over consumption is a CRRA specification with a coefficient of relative risk aversion $\sigma = 2$, which is consistent with Conesa et al. (2009) and Aiyagari and McGrattan (1998). Disutility over labor exhibits a constant intensive margin Frisch elasticity. We choose $\gamma = 0.5$ such that the Frisch elasticity consistent with the majority of the related literature as well as the estimates in Kaplan (2012).

We calibrate the labor disutility parameter χ_1 so that the cross sectional average of hours is one third of the time endowment. Finally, χ_2 is a fixed utility cost of earning labor income before retirement. The fixed cost generates an extensive margin decision through a non-convexity in the utility function. We choose χ_2 to match the empirical observation that seventy percent of the population has retired by the normal retirement age.

Labor Productivity Process: We take the labor productivity process from the estimates in Kaplan (2012) based on the estimates from the PSID data.[9] The deterministic labor productivity profile, $\{\theta_j\}_{j=1}^{\bar{J}_{ret}}$, is (i) smoothed by fitting a quadratic function in age, (ii) normalized such that the value equals unity when an agent enters the economy, and (iii) extended to cover ages 21 through 70 which we define as the last period in which agents are assumed to be able to participate in the labor activities ($\bar{J}_{ret}$).[10] The permanent, persistent, and transitory idiosyncratic shocks to individual's productivity are normally distributed with zero mean. The remaining parameters are also set in accordance with the Kaplan's (2012) estimates: $\rho = 0.958$, $\sigma_\kappa^2 = 0.065$, $\sigma_\nu^2 = 0.017$ and $\sigma_\epsilon^2 = 0.081$.

Government: Consistent with Aiyagari and McGrattan (1998) we set government debt equal to two-thirds of output. We set government consumption equal to 15.5 percent of output consistent. This ratio corresponds to the average of government expenditures to GDP from 1998 through 2007.[11]

Income Taxation: The income tax function and parameter values are from

[9]For details on estimation of this process, see Appendix E in Kaplan (2012).

[10]The estimates in Kaplan (2012) are available for ages 25-65.

[11]We exclude government expenditures on Social Security since they are explicitly included in our model.

Gouveia and Strauss (1994). The functional form is:

$$\Upsilon(y) = \tau_0 \left(y - \left(y^{-\tau_1} + \tau_2 \right)^{-\frac{1}{\tau_1}} \right)$$

The authors find that $\tau_0 = 0.258$ and $\tau_1 = 0.768$ closely match the U.S. tax data. When calibrating the model we set τ_2 such that the government budget constraint is satisfied.

Social Security: We set the normal retirement age to 66. Consistent with the minimum and maximum retirement ages in the U.S. Social Security system, we set the interval in which agents can retire to the ages 62 and 70. The early retirement penalty and later retirement credits are set in accordance with the Social Security program. In particular, if agents retire up to three years before the normal retirement age agents benefits are reduced by 6.7 percent for each year they retire early. If they choose to retire four or five years before the normal retirement age benefits are reduced by an additional 5 percent for these years. If agents choose to delay retirement past normal retirement age then their benefits are increased by 8 percent for each year they delay. The marginal replacement rates in the progressive Social Security payment schedule ($\tau_{r1}, \tau_{r2}, \tau_{r3}$) are also set at their actual respective values of 0.9, 0.32 and 0.15. The bend points where the marginal replacement rates change (b_1^{ss}, b_2^{ss}, b_3^{ss}) and the maximum earnings ($\bar{x}$) are set equal to the actual multiples of mean earnings used in the U.S. Social Security system so that b_1^{ss}, b_2^{ss} and $b_3^{ss} = \bar{x}$ occur at 0.21, 1.29 and 2.42 times average earnings in the economy. We set the payroll tax rate, τ_{ss} such that the program's budget is balanced. In our baseline model the payroll tax rate is 10.3 percent, roughly equivalent with the statutory rate.[12]

Infinitely Lived Agent Model: The infinitely lived agent model does not have a age-dependent wage profile. For comparability across models, we replace the age-dependent wage profile with the population-weighted average θ_j.[13]

[12] Although the payroll tax rate in the U.S. economy is slightly higher than our calibrated value, the OASDI program includes additional features outside of the retirement benefits.

[13] When calibrating the stochastic process for idiosyncratic productivity shocks, we use the same process in the both the life cycle and infinitely lived agent models. Using the same

Table 1: Calibration Targets and Parameters for Baseline Economy.

Description	Parameter	Value	Target or Source
Demographics			
Maximum Age	J	100	By Assumption
Min/Max Retirement Age	$\underline{J}_{ret}, \bar{J}_{ret}$	62, 70	Social Security Program
Population Growth	g_n	1.1%	Conesa et al (2009)
Survival Rate	$\{s_j\}_{j=1}^{J}$	—	Bell and Miller (2002)
Preferences and Borrowing			
Coefficient of RRA	σ	2.0	Kaplan (2012)
Frisch Elasticity	γ	0.5	Kaplan (2012)
Coefficient of Labor Disutility	χ_1	55.3	Avg. Hours Worked = 1/3
Fixed Utility Cost of Labor	χ_2	1.038	70% retire by NRA
Discount Factor	β	1.012	Capital/Output = 2.7
Borrowing Limit	$\underline{a}$	0	By Assumption
Technology			
Capital Share	α	0.36	NIPA
Capital Depreciation Rate	δ	0.0833	Investment/Output = 0.255
Productivity Level	Z	1	Normalization
Output Growth	g_y	1.85%	NIPA
Labor Productivity			
Persistent Shock, autocorrelation	ρ	0.958	Kaplan (2012)
Persistent Shock, variance	σ_{ν}^2	0.017	Kaplan (2012)
Permanent Shock, variance	σ_{κ}^2	0.065	Kaplan (2012)
Transitory Shock, variance	σ_{ϵ}^2	0.081	Kaplan (2012)
Mean Earnings, Age Profile	$\{\theta\}_{j=1}^{\bar{J}_{ret}}$	—	Kaplan (2012)
Government Budget			
Government Consumption	G/Y	0.155	NIPA Average 1998-2007
Government Savings	B/Y	-0.667	NIPA Average 1998-2007
Marginal Income Tax	τ_0	0.258	Gouveia and Strauss (1994)
Income Tax Progressivity	τ_1	0.786	Gouveia and Strauss (1994)
Income Tax Progressivity	τ_2	4.541	Balanced Budget
Social Security			
Payroll Tax	τ_{ss}	0.103	Social Security Program
SS Replacement Rates	$\{\tau_{ri}\}_{i=1}^{3}$	See Text	Social Security Program
SS Replacement Bend Points	$\{b_i^{ss}\}_{i=1}^{3}$	See Text	Social Security Program
SS Early Retirement Penalty	$\{\kappa_i\}_{i=1}^{3}$	See Text	Social Security Program

In the absence of a retirement decision, we set $\chi_2 = 0$. Lastly, we recalibrate

underlying process will imply that cross-sectional wealth inequality will be different across the two models. One reason is that the life cycle model will have additional cross-sectional inequality due to the humped shaped savings profiles, which induces the building, stationary, and deaccumulation phases. We view these difference in inequality as a fundamental difference between the two models and, therefore, choose not to specially alter the infinitely lived agent model to match a higher level of cross-sectional inequality.

the parameters (β, χ) to the same targets as in the life cycle model and choose τ_2 to balance the government's budget.

5 Quantitative Model Comparisons

Having described how we use external data to discipline the models' structural parameters, we use the calibrated model to measure optimal policy across the life cycle and infinitely lived agent models. Then we perform a series of counterfactual experiments to highlight the mechanisms that generate differences in optimal policy across the models.

5.1 Optimal Public Policy

The government maximizes social welfare by choosing a budget feasible level of public savings, B, subject to allocations being a competitive equilibrium. We consider an ex-ante Utilitarian social welfare criterion that evaluates the expected lifetime utility of an agent that has yet to enter the steady state economy.[14] For the life cycle model, the government's welfare maximization problem is:

$$S(V_1, \lambda_1) \equiv \max_B \int V_1(a, \varepsilon, x, \zeta; B)\, d\lambda_1(a, \varepsilon, x, \zeta; B)$$

$$\text{s.t.} \quad (2),\ (3)$$

where the value function $V_1(\cdot; B)$, distribution function $\lambda_1(\cdot; B)$ and policy functions embedded in equation (2) and equation (3) are determined in competitive equilibrium and depend on the government's choice of public savings. Since the distribution of taxable income and tax revenues depend on public savings, we adjust the Social Security payroll tax rate τ_{ss} to ensure that Social Security is self-financing and, furthermore, adjust the income tax

[14] Our analysis focuses on welfare across steady states. This analysis omits the transitional costs between steady states which can be large. See Domeij and Heathcote (2004), Fehr and Kindermann (2015) and Dyrda and Pedroni (2016).

parameter τ_0 to ensure that the government budget is balanced.[15]

For the infinitely lived agent model, the government's welfare maximization problem is:

$$S_\infty(V,\lambda) \equiv \max_B \left\{ \int V(a,\varepsilon;B)\ d\lambda(a,\varepsilon;B) \quad \text{s.t.} \quad G = rB + \Upsilon_y(\tau_0,B) \right\}$$

The infinitely lived agent model government's welfare maximization problem is nearly identical to that of the life cycle model's, except that the value function and distribution function do not depend on age and there is no Social Security program, so that equation (3) does not define the feasible set.

We find that the two models generate starkly different optimal policies, which are reported in Table 2. In the infinitely lived agent model, the government optimally holds debt equal to 22% of output.[16] In the life cycle model, on the other hand, the government optimally holds savings equal to 59% of output. Thus, including life cycle features causes optimal policy to switch from public debt to savings, with an 80 percentage point swing in optimal policy.

5.2 Welfare Decomposition

While the infinitely lived agent model prescribes that the government hold public debt, the life cycle model's optimal policy prescribes holding public savings. What is the welfare loss from incorrectly implementing a public debt policy?

We quantify the welfare consequence of ignoring life cycle features and, as a consequence, adopting a public debt instead of a public savings policy. To do so, suppose that the government implements the optimal debt policy

[15] We choose to use τ_0 to balance the government budget instead of the other income taxation parameters (τ_1, τ_2) so that the average income tax rate is used to clear the budget, as opposed to changing in the progressivity of the income tax policy. The average tax rate is the closest analogue to the flat tax that Aiyagari and McGrattan (1998) use to balance the government's budget in their model.

[16] This is generally consistent with Aiyagari and McGrattan's (1998) optimal policy. This paper assumes a different stochastic process governing labor productivity, a different utility function, non-linear income taxation and parameter values. A quantitative decomposition of these model differences are available upon request.

Table 2: Prices and Aggregates Across Models

	Life Cycle			Infinitely Lived		
	Base.	**Opt.**	**% Δ**	**Base.**	**Opt.**	**% Δ**
Public Savings/Output	-0.67	0.59	189.0	-0.67	-0.22	67.5
Private Savings/Output	3.37	2.42	-28.2	3.35	2.96	-11.7
Capital/Output	2.70	3.01	11.5	2.70	2.74	1.9
Output	0.93	1.01	8.6	1.16	1.17	1.5
Labor	0.53	0.54	2.2	0.66	0.67	0.4
Interest Rate	5.0%	3.6%	-1.4	5.0%	4.8%	-0.2
Wage	1.12	1.19	6.3	1.12	1.13	1.0

from an infinitely lived agent economy when the true economy is a life cycle economy. We then measure the welfare loss from implementing a suboptimal debt policy using *consumption equivalent variation (CEV)*. CEV is the percent of lifetime consumption that an agent would be willing to pay, ex ante, in order to live in an economy with an optimal public savings policy instead of a suboptimal public debt policy.

Table 3 reports the consumption equivalent variation. We find that an 80 percentage point difference in fiscal policy corresponds to a welfare loss of 0.42% of expected lifetime consumption. The welfare loss is economically significant, demonstrating that ignoring life cycle features when determining optimal debt policy will have nontrivial welfare effects. The same 80 percentage point change to government policy in the infinitely lived agent model leads to much smaller welfare effects. In particular, an infinitely lived agent would only sacrifice 0.04% of lifetime consumption in order to live in the economy in which the government holds optimal debt instead of 59% of output in public savings.

The welfare gains from implementing optimal policy reflect the government's desire to improve the aggregate resources available to agents and the allocation of those resources across agents. In order to characterize these welfare effects, we decompose the consumption equivalent variation (denoted

Table 3: Welfare Decompositions

(% Change)	**Life Cycle**	**Infinitely Lived**
Overall CEV	0.42	-0.04
Level (Δ_l)	0.92	-0.70
Consumption (Δ_{C_l})	1.36	0.90
Hours (Δ_{H_l})	-0.43	-1.58
Distribution (Δ_d)	-0.50	0.66
Consumption (Δ_{C_d})	0.08	-0.20
Hours (Δ_{H_d})	-0.58	0.86

The Life Cycle Model welfare decomposition compares allocations under a -22% public debt-to-output and the optimal 59% public savings-to-output ratio. The Infinitely Lived Agent Model welfare decomposition compares allocations under the optimal -22% public debt-to-output and a 59% public savings-to-output ratio. The Level and Distribution decompositions are given by $100((1+\Delta_{C_l})(1+\Delta_{H_l})-1)$ and $100((1+\Delta_{C_d})(1+\Delta_{H_d})-1)$.

Δ_{CEV}) into a *level effect* (Δ_l) and a *distribution effect* (Δ_d) as follows:[17]

$$(1+\Delta_{CEV}) = [\underbrace{(1+\Delta_{C_l})(1+\Delta_{H_l})}_{\equiv(1+\Delta_l)}] \cdot [\underbrace{(1+\Delta_{C_d})(1+\Delta_{H_d})}_{\equiv(1+\Delta_d)}].$$

The level effect measures the average agent's change in welfare as a result of changes in aggregate consumption (Δ_{C_l}) and aggregate hours (Δ_{H_l}). The level effect captures the welfare change for a fictitious "representative agent," absent distributional concerns of policy. On the other hand, the distribution effect measures the remaining change in welfare that results from a change in the distribution of consumption (Δ_{C_d}) and hours (Δ_{H_d}) across agents.[18]

[17]More generally, we follow Floden (2001) in characterizing four components of the CEV: a level effect (Δ_L), an insurance effect (Δ_I), a redistribution effect (Δ_R) and a labor hours effect (Δ_H). We combine the insurance and redistribution effects to form the "distribution effect". Appendix A.3 formally derives the decomposition.

[18]Note that the life cycle model only assumes initial heterogeneity with respect to the permanent and transitory components of labor productivity, but not initial wealth heterogeneity. While allowing for heterogeneity in the initial wealth distribution could generate a larger distribution effect in welfare changes, the PSID and SCF document low levels and relatively small dispersion in individuals' wealth upon entering the labor market.

Adopting public savings has differential welfare effects across models. Table 3 reports that the 0.42% welfare improvement from adopting public savings in the life cycle model can be decomposed into a 0.92% increase in the level effect and a partially offsetting 0.50% decrease in the distribution effect. The opposite holds for adopting public savings in the infinitely lived agent model, where the 0.04% welfare loss corresponds to a 0.70% decrease in the level effect and 0.66% increase in the distribution effect. These differences in level and distribution effects reflect the varying welfare impact of competing mechanisms across models.

The level effect reflects a difference in the efficacy of the insurance channel across models. To see this, first note that adopting public savings induces a higher wage and a lower interest rate. The higher wage encourages additional labor hours, which increases the resources available for agents' consumption (as seen in Figure 2 for the life cycle model) but worsens total disutility from hours worked. However, Table 3 reports that the percent change in utility from increased aggregate consumption is higher in the life cycle model (1.36) than in the infinitely lived agent model (0.90), despite a larger percent increase in labor disutility in the latter. To account for the relatively smaller consumption increase in the infinitely lived agent model, recall that the lower interest rate discourages private savings in each model. In the infinitely lived agent model, however, the policy also reduces *ex ante* average wealth. This is because, by the nature of living infinite lifespans, aggregate savings is equivalent to ex ante wealth. Therefore, the lower interest rate worsens an infinitely lived agent's ex ante self-insurance and a larger fraction of the population must sacrifice consumption due to binding liquidity constraints.[19] In contrast, the lower interest rate has no effect on initial allocations in the life cycle model because initial wealth is zero and does not respond to prices.[20] We provide quantitative evidence for this mechanism in Section 5.3.1.

On the other hand, the distribution effect reflects differences in the inequality channel across models. Again, adopting public savings corresponds

[19]Relative to Floden (2001), the effect of policy on insurance is usually defined as a distribution effect. However, in order to compare model outcomes, we measure it as an average effect on consumption an hours that is captured by the level effect.

[20]The lower interest rate may decreases average savings across the lifetime, however it does not have as large of a welfare effect since it does not change the initial distribution.

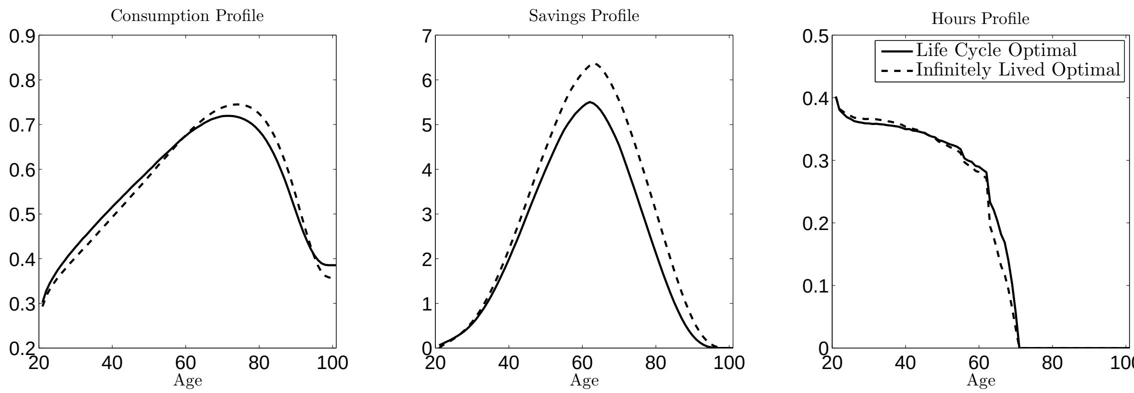


Figure 2: Solid lines are cross-sectional averages for consumption, savings, and hours by age in the life cycle economy under its optimal public savings policy. The dashed lines are cross-sectional averages for the suboptimal debt policy from the infinintely lived agent economy.

to a higher wage and lower interest rate. In the life cycle model, labor income makes up a relatively larger share of an average agent's income and a higher wage especially encourages agents with high labor productivity to work more at the margin. Therefore, adopting public savings creates more total income inequality. In the infinitely lived agent model, on the other hand, asset income is a relatively larger share of an average agent's income. A lower interest rate discourages savings and therefore decreases wealth inequality, as well as total income inequality. Insofar as agents are risk averse, agents dislike inequality because it induces greater uncertainty over the set of possible ex post allocations. Thus, the negative distribution effect in the life cycle model reflects greater income inequality while the positive distribution effect in the infinitely lived agent model reflects lower income inequality. We provide quantitative evidence for this mechanism in Section 5.3.2.

5.3 Life Cycle Features

There are three main differences between the life cycle and infinitely lived agent models: (i) agents in the life cycle model experience an accumulation phase while agents in the infinitely lived agent model experience a perpetual stationary phase, (ii) age-dependent features in the life cycle model, such as mortality risk, an age-dependent wage profile, retirement and Social Security, and (iii) lifespan. We begin by demonstrating that the introduction of the accumulation phase in the life cycle model predominantly explains the differences in optimal policies between the two models. Then we systematically decompose the effects of each of the three model differences on optimal policy.

5.3.1 The Accumulation Phase

This section quantifies the importance of the accumulation phase for explaining the difference in optimal policies between the life cycle and infinitelylived agent models. We do this by constructing an approximation to the infinitely lived agent economy that features an accumulation phase. Relative to the infinitely lived agent model, the counterfactual model mainly differs from the infinitely lived agent model in that agents are endowed with zero wealth. In order to activate the accumulation phase we assume agents have finite lifespans. However, we assume that agents die at the end of $J = 1000$ periods, a sufficiently large terminal age to mimic the infinitely lived agent model. Therefore, by construction, the fundamental difference between the counterfactual model and the infinitely lived agent model is the accumulation phase.[21]

Using the calibrated counterfactual model, we conduct a computational experiment to isolate the impact of the accumulation phase on optimal policy. Suppose that the government chooses policy according to an alternative social welfare criterion that places less weight on the flow of utility during youth than does the ex ante Utilitarian welfare criterion. In particular, suppose that the alternative social welfare criterion only incorporates the expected present value of utility after a given age $j^* > 1$, and ignores the flow of utility from ages 1 to $j^* - 1$. Government policy, therefore, maximizes agents' expected utility as of age j^*, subject to allocations being determined in competitive

[21] Neither the infinitely lived nor the counterfactual model feature any age-dependent features (e.g., no mortality risk, no age-dependent wage profile, no retirement and no Social Security). In order to make quantitative comparisons across models, the counterfactual model's parameters are recalibrated to match all relevant the targets described in Section 4.

equilibrium:

$$\tilde{S}(V_{j^*}, \lambda_{j^*}) \equiv \max_B \left\{ \int V_{j^*}(a, \varepsilon; B)\, d\lambda_{j^*}(a, \varepsilon; B) \quad \text{s.t.} \quad G = rB + \Upsilon_y(\tau_0, B) \right\}.$$

Figure 3 plots the optimal policy under this alternative welfare criterion as a function of threshold age, j^*. We observe that optimal policy monotonically decreases from the public savings to output ratio of 2.35 when $j^* = 21$ to an optimal debt policy when $j^* \geq 71$, or when the government ignores at least 5.2% of agents' early lifetime. Across models, higher public debt (lower public savings) crowds out the productive capital stock and leads to a higher interest rate. The higher interest rate encourages agents to save, which improves self insurance. When the age threshold is small, the government includes agents' utility during the accumulation phase in its welfare maximization calculation. As a result, agents' welfare improvement from accumulating more precautionary savings is offset by the utility cost of accumulating that savings. However, when the age threshold is large, the government ignores the flow of utility for agents in the accumulation phase and there is only a large welfare improvement from encouraging more self insurance. The right panel of Figure 3 shows that ignoring at least 5.2% of agents' early lifetime corresponds to ignoring at least 69% of the accumulation phase.

The experiment shows that the existence of an accumulation phase is crucial to the optimality of public savings. Without the accumulation phase, the benefits of the insurance channel strengthen and lead to the optimality of public debt.

5.3.2 Decomposing the Effects of Life Cycle Features

Apart from the accumulation phase, there are two remaining differences between the life cycle and infinitely lived agent models: lifespan and age-dependent features (e.g., mortality risk, age-dependent wage profile, retirement and Social Security). Next, we quantify the effect of each of these differences on optimal policy. Unlike removing the accumulation phase, we find that removing lifespan and age-dependent features shifts optimal policy towards *more* public savings.

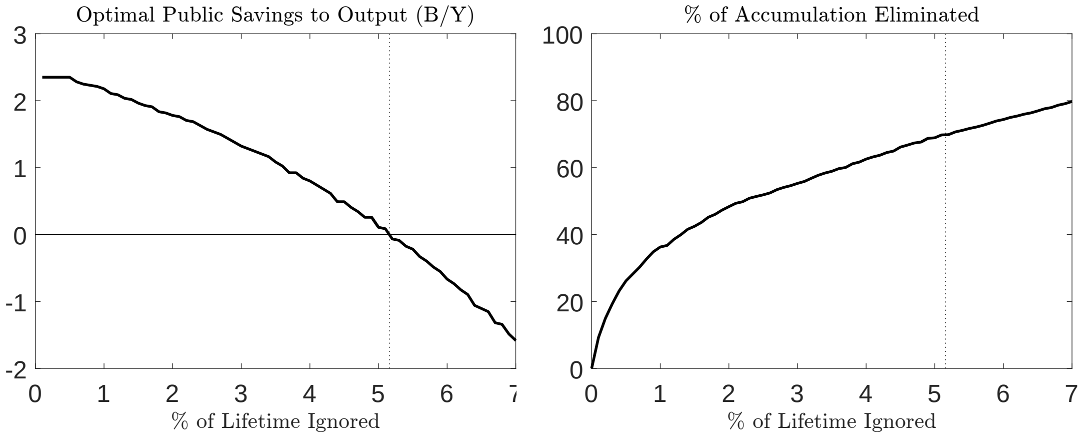


Figure 3: The left panel graphs the optimal public savings to output ratio (y-axis) associated with ignoring a given percent of early life utility flows (x-axis). The percent of "Lifetime Ignored" is measured as $100 \cdot (j^*/J)$, using the given value of j^* and $J = 1000$. The right panel graphs the percent of accumulation that is eliminated under the optimal policy associated with ignoring a given percent of early life utility flows. The percent of eliminated wealth accumulation is defined as the average private savings of j^*-age agents relative to the peak average savings and converted to a percent, given a particular optimal public savings policy. The vertical dashed line demarcates the percent of early lifetime utility ignored at which optimal policy switches from public savings to debt.

In order to characterize the individual effects of these differences on optimal policy between the life cycle and infinitely lived agent models, we construct two counterfactual economies. The first is the "No Age-Dependent Features" economy, which is a version of the life cycle model that excludes all age-dependent features (e.g., no mortality risk, no age-dependent wage profile, no retirement and no Social Security) while maintaining the lifespan of $J = 81$ periods. The second is the "Long Life" economy, which removes age-dependent model features and also extends agents' lifetime to $J = 401$ periods.[22]

Table 4 reports the optimal in the benchmark life cycle, infinitely lived, and counterfactual models. First, comparing the baseline life cycle model and "No Age-Dependent Features" economy isolates the effect of age-dependent features, which leads to an increase in the working lifetime due to the removal of retirement and mortality. We find that the optimal savings-to-output shifts

[22]In order to make quantitative comparisons across models, each counterfactual model's parameters are recalibrated to match all relevant the targets described in Section 4.

Table 4: Optimal Public Savings-to-Output Ratios

	Counterfactuals		
Life Cycle	**No Age Features (81 periods)**	**Long Lifespan (401 periods)**	**Infinitely Lived**
0.59	2.00	2.48	-0.22

from 59% to 200%. Comparing the "No Age-Dependent Features" and "Long Life" counterfactual economies isolates the effect of further increasing agents' working lifetime and lifespan. This effect additionally increases the optimal savings-to-output from 200% to 248%. Finally, comparing the "Long Life" economy with the infinitely lived agent model highlights the effect of the accumulation phase on optimal policy. In contrast, eliminating the accumulation phase causes optimal policy to switch from savings-to-output of 59% to debt-to-output of 22% in the infinitely lived agent model.

The main mechanism by which removing age-dependent features and lengthening agents' lifespan increases optimal public savings is an extension of the span of life that agents spend working. In the life cycle model, there is a tendency for wealth inequality to increase with an extension of agents' expected working lifespan and, in turn, this generates a greater amount of inequality in lifetime asset income. Table 5(a) reports that, indeed, measures of lifetime asset income inequality (the Gini coefficient and the coefficient of variation), increase under the baseline calibration when these two features are removed.[23] Likewise, the cumulative distribution functions for savings under the baseline policy (see Figure 4) demonstrate that wealth becomes more unequal when either of these features are removed. In contrast, Table 5(b) demonstrates that *total* lifetime income inequality tends to decrease when age-dependent features are removed and the lifespan is extended. Therefore, interest income becomes a larger source of overall income inequality when these features are removed.

[23] To construct inequality measures, we use lifetime asset income as a share of lifetime total income: $\sum_{j=1}^{J} s_j \left(\frac{1}{1+r}\right)^{j-1} ra_j \;/\; \left(\sum_{j=1}^{J} s_j \left(\frac{1}{1+r}\right)^{j-1} we_j h_j + \sum_{j=1}^{J} s_j \left(\frac{1}{1+r}\right)^{j-1} ra_j\right)$. For the No Age-Dependent Features and Long Lifespan counterfactual models, there is no mortality risk so that $s_j = 1$ for all $j = 1, \ldots, J$.

Table 5: Income Composition and Inequality

	Life Cycle	Counterfactuals: No Age Features (81 periods)	Counterfactuals: Long Lifespan (401 periods)
(a) Asset Income Inequality in Baseline Calibration			
Coefficient of Variation	0.33	0.34	0.49
Gini Coefficient	0.19	0.20	0.28
(b) Lifetime Total Income Inequality			
Coefficient of Variation			
Baseline	0.36	0.32	0.31
Optimal	0.35	0.30	0.27
% Change	-1.8%	-7.1%	-13.8%
Gini Coefficient			
Baseline	0.20	0.18	0.17
Optimal	0.19	0.16	0.15
% Change	-2.0%	-6.9%	-12.0%

Because of risk aversion, agents dislike inequality and thus policy has a role to improve welfare by reducing this income inequality. In the life cycle and counterfactual models, moving from public debt to public savings increases the wage and decreases the interest rate. All else equal, this increases lifetime income inequality from savings and decreases lifetime income inequality from labor earnings. Thus, optimal policy must weigh this tradeoff. Asset income contributes more to lifetime total income inequality when age-dependent features are removed and lifespan is extended. Accordingly, shifting toward a higher level of public savings will reduce lifetime total income inequality. The change in the wealth distribution in Figure 4 and the total lifetime income inequality measures in Table 5(b) demonstrate that, in fact, adopting an optimal public savings policy reduces both lifetime asset income and total income inequality. Thus, overall, eliminating age-dependent features and extending the lifespan both cause an increase in the optimal level

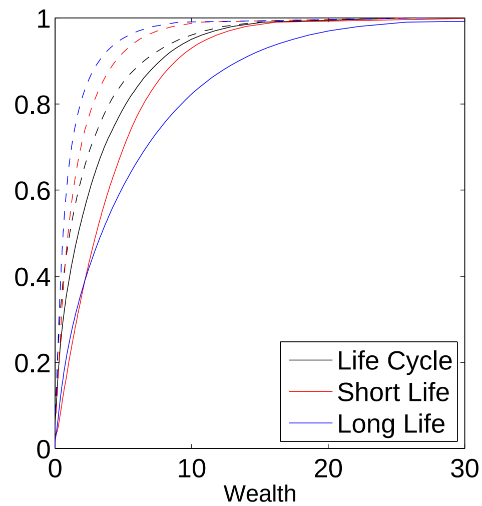


Figure 4: Cumulative Distribution Function for Wealth. Solid lines represent the baseline economy and dashed lines represent economies with optimal policy.

of public savings due to the inequality channel.

Finally, the primary difference between the Long Life counterfactual model and the infinitely lived agent model is the existence of an accumulation phase. Despite their other common features (e.g., no age-dependent features, and long or infinite lifetimes), the infinitely lived agent model features a starkly different optimal policy of public debt as opposed to public savings. As Section 2 explained, the existence of an accumulation phase mitigates the efficacy of the insurance channel while extending agents' working lifetime further enforces the inequality channel. Thus, when comparing the life cycle and the infinitely lived agent models, the existence of age-dependent features and a shorter lifespan drive optimal policy toward public debt while the existence of the accumulation phase drives optimal policy toward public savings. Overall, we find that the effects of the accumulation phase dominate the effects of other life cycle model features on optimal policy, thereby ultimately resulting in the optimality of public savings.

6 Conclusion

This paper measured the optimal quantity of public debt in a variant of the incomplete markets model that allows for an explicit life cycle. We find that it is optimal for the government to hold *savings* equal to 59% of output when life cycle features are included. In contrast, we find that it is optimal for the government to hold *debt* equal to 22% of output when these life cycle features are excluded. Furthermore, there are economically significant welfare consequences from not accounting for life cycle features when determining the optimal policy. We find that if a government implemented the infinitely lived agent model's optimal 20% debt-to-output policy in the life cycle model, then life cycle agents would be worse off by nearly one-half percent of expected lifetime consumption.

The substantial difference in optimal policies across the two models is primarily due to differences in the effectiveness and benefit of public debt encouraging agents to hold precautionary savings. Generally, higher government debt (or decreasing government savings) tends to crowd out the stock of productive capital, and leads to a higher interest rate which encourages agents to hold more savings. However, this channel is significantly less beneficial in the life cycle model relative to the infinitely lived agent model. This is because, agents in the infinitely lived agent model do not experience an accumulation phase but instead experience a perpetual stationary phase in which agents have accumulated a target level of assets, around which savings fluctuates. If the government holds more public debt, then the *steady state* level of aggregate savings is larger and the average agent has more wealth *ex ante*. In contrast, life cycle agents enter the economy with zero wealth and immediately begin an accumulation phase, in which agents build wealth for precautionary reasons and to finance post-retirement consumption. Thus, although changes in the interest rate may increase the level of savings in the stationary phase for life cycle agents, these agents' initial wealth will not respond to policy and agents still need to accumulate this wealth during the first phase of their lifetimes. Ultimately, this significantly reduces the benefit of government debt in the life cycle model.

When using quantitative models to answer economic questions, economists are constantly faced with a trade-off between tractability and realism. Our results demonstrate that when examining the welfare consequences of public debt or savings, it is not without loss of generality to utilize the more tractable infinitely lived agent model instead of a life cycle model.

References

Aiyagari, S. R. and E. R. McGrattan (1998): "The optimum quantity of debt," *Journal of Monetary Economics*, 42, 447–469.

Bell, F. and M. Miller (2002): "Life Tables for the United States Social Security Area 1900 - 2100," Office of the Chief Actuary, Social Security Administration, Actuarial Study 116.

Carroll, C. D. (1992): "The Buffer-Stock Theory of Saving: Some Macroeconomic Evidence," *Brookings Papers on Economic Activity*, 23, 61–156.

——— (1997): "Buffer-Stock Saving and the Life Cycle/Permanent Income Hypothesis," *Quarterly Journal of Economics*, 112, 1–55.

Chaterjee, S., J. Gibson, and F. Rioja (2016): "Optimal Public Debt Redux," Unpublished.

Conesa, J. C., S. Kitao, and D. Krueger (2009): "Taxing Capital? Not a Bad Idea after All!" *American Economic Review*, 99, 25–48.

Dávila, J., J. Hong, P. Krusell, and J.-V. Ríos-Rull (2012): "Constrained Efficiency in the Neoclassical Growth Model With Uninsurable Idiosyncratic Shocks," *Econometrica*, 80, 2431–2467.

Desbonnet, A. and T. Weitzenblum (2012): "Why Do Governments End Up with Debt? Short-Run Effects Matter," *Economic Inquiry*, 50, 905–919.

Domeij, D. and J. Heathcote (2004): "On The Distributional Effects Of Reducing Capital Taxes," *International Economic Review*, 45, 523–554.

Dyrda, S. and M. Pedroni (2016): "Optimal fiscal policy in a model with uninsurable idiosyncratic shocks," Unpublished.

Fehr, H. and F. Kindermann (2015): "Taxing capital along the transition - Not a bad idea after all?" *Journal of Economic Dynamics and Control*, 51, 64–77.

Floden, M. (2001): "The effectiveness of government debt and transfers as insurance," *Journal of Monetary Economics*, 48, 81–108.

Fuster, L., A. Imrohoroglu, and S. Imrohoroglu (2008): "Altruism, incomplete markets, and tax reform," *Journal of Monetary Economics*, 55, 65–90.

Gouveia, M. and R. Strauss (1994): "Effective Federal Individual Income Tax Functions: An Exploratory Empirical Analysis," *National Tax Journal*, 47, 317–339.

Kaplan, G. (2012): "Inequality and the life cycle," *Quantitative Economics*, 3, 471–525.

Röhrs, S. and C. Winter (2016): "Reducing Government Debt in the Presence of Inequality," Unpublished.

A Appendix

A.1 Pecuniary Externality

[To Be Completed.]

A.2 Construction of the Balanced Growth Path

We construct the Balanced Growth Path in multiple parts. First we construct the Balanced Growth Path using aggregates from the models. Then, we construct the Balanced Growth Path using individual agents' allocations. The last two sections develop the Balanced Growth Path for any features unqiue to the infinitely lived agent or life cycle models.

A.2.1 Aggregate Conditions

Balanced Growth Path: A Balanced Growth Path (BGP) is a sequence

$$\{C_t, A_t, Y_t, K_t, L_t, B_t, G_t\}_{t=0}^{\infty}$$

such that (i) for all $t = 0, 1, \ldots$ $C_t, A_t, Y_t, K_t, B_t, G_t$ grow at a constant rate g,

$$\frac{Y_{t+1}}{Y_t} = \frac{C_{t+1}}{C_t} = \frac{A_{t+1}}{A_t} = \frac{K_{t+1}}{K_t} = \frac{B_{t+1}}{B_t} = \frac{G_{t+1}}{G_t} = 1 + g$$

(ii) per capita variables all grow at the same constant rate g_w:

$$\frac{Y_{t+1}/N_{t+1}}{Y_t/N_t} = \frac{C_{t+1}/N_{t+1}}{C_t/N_t} = \frac{A_{t+1}/N_{t+1}}{A_t/N_t} = \frac{K_{t+1}/N_{t+1}}{K_t/N_t} = \frac{B_{t+1}/N_{t+1}}{B_t/N_t} = \frac{G_{t+1}/N_{t+1}}{G_t/N_t} = 1 + g_w$$

and (iii) hours worked per capita are constant:

$$\frac{L_{t+1}}{N_{t+1}} = \frac{L_t}{N_t} = \frac{L_0}{N_0}$$

Denote time 0 variables without a time subscript, for example $L \equiv L_0$.

Growth Rates: Let growth derive from TFP $g_z > 0$ and population $g_n > 0$.

Then on a balanced growth path we assume:

$$Z_t = (1+g_z)^t Z$$

$$N_t = (1+g_n)^t N$$

where z and N are steady state values. In steady state $Y = ZK^\alpha L^{1-\alpha}$. Let output growth be given by $g > 0$. Therefore the production function gives:

$$Y_t = Z_t K_t^\alpha L_t^{1-\alpha}$$

$$(1+g) = (1+g_z)^{\frac{1}{1-\alpha}}(1+g_n)$$

Lastly, from parts (ii) and (iii) of the Balanced Growth Path definition, we can solve for the growth of per capita variables:

$$\begin{aligned}
\frac{Y_{t+1}/N_{t+1}}{Y_t/N_t} &= \frac{Z_{t+1}}{Z_t}\left(\frac{K_{t+1}/N_{t+1}}{K_t/N_t}\right)^\alpha \left(\frac{L_{t+1}/N_{t+1}}{L_t/N_t}\right)^{1-\alpha} \\
(1+g_w) &= (1+g_z)^{\frac{1}{1-\alpha}}
\end{aligned}$$

where the growth in labor is:

$$\frac{L_{t+1}}{L_t} = \frac{L_{t+1}/N_{t+1}}{L_t/((1+g_n)N_t)} = 1 + g_n$$

Prices: From Euler's theorem we know:

$$Y_t = \alpha Y_t + (1-\alpha)Y_t = (r_t + \delta)K_t + w_t L_t$$

Accordingly, the wage and interest rate depend on the capital-labor ratio. Growth in the capital-labor ratio is:

$$\frac{K_{t+1}/L_{t+1}}{K_t/L_t} = (1+g_z)^{\frac{1}{1-\alpha}} = 1 + g_w$$

Therefore, the growth rate for the wage is:

$$\frac{w_{t+1}}{w_t} = \frac{Z_{t+1}}{Z_t} \cdot \left(\frac{K_{t+1}/L_{t+1}}{K_t/L_t} \right)^{\alpha} = 1 + g_w$$

and the growth rate for the interest rate is:

$$\frac{r_{t+1} + \delta}{r_t + \delta} = \frac{Z_{t+1}}{Z_t} \cdot \left(\frac{K_{t+1}/L_{t+1}}{K_t/L_t} \right)^{\alpha - 1} = 1$$

Therefore wages grow while interest rates do not.

Equilibrium Conditions: We will now derive the detrended asset market clearing condition, resource constraint, government budget constraint. The *asset market clearing condition* is:

$$K_t = A_t + B_t \quad \Longrightarrow \quad K = A - B$$

The *resource constraint* is:

$$C_t + K_{t+1} + G_t = Y_t + (1 - \delta) K_t \quad \Longrightarrow \quad C + (g + \delta) K + G = Y$$

and the *government budget constraint* is:

$$G_t + r B_t = T_t + B_{t+1} - B_t \quad \Longrightarrow \quad G + (r - g) B = T$$

A.2.2 Individual Conditions

Preferences: We assume that labor disutility has a time-dependent component. Specifically, we assume labor disutility grows at the same rate as the utility over consumption, such that $v_{t+1}(h) = (1 + g_w)^{1-\sigma} v_t(h)$. Therefore, total utility is:

$$U_t(c_t, h_t) = u(c_t) - v_t(h_t) = \left[(1 + g_w)^{1-\sigma} \right]^t (u(c) - v(h)) .$$

Social Security: In order for the AIME to grow at the same rate as the wage, we assume a cost of living adjustment (COLA) on Social Security taxes and payments. For social security taxes, the cap on eligible income grows at the rate of wage growth, $\bar{x}_t = (1+g_w)^t \bar{x}$. Furthermore, base payment bend points $b^{ss}_{i,t} = (1+g_w)^t b^{ss}_i$ and base payment values $\tau_{r,i,t} = (1+g_w)^t \tau_{r,i}$ for $i = 1, 2, 3$.

Tax Function: On a Balanced Growth Path, (c_t, a'_{t+1}, a_t) and $\tilde{y}_t$ must all grow at the same rate as the wage. Furthermore, the tax function must grow at the same rate as the wage. Recall the tax function:

$$\Upsilon(\tilde{y}_t) = \tau_0 \left(\tilde{y}_t - \left(\tilde{y}_t^{-\tau_1} + \tau_2 \right)^{-\frac{1}{\tau_1}} \right)$$

therefore, τ_2 must grow at the same rate as $\tilde{y}_t$. Rewrite as:

$$\Upsilon(\tilde{y}_t) = \tau_0 \left((1+g_w)^t \tilde{y} - \left([(1+g_w)^t]^{-\tau_1} \tilde{y}^{-\tau_1} + [(1+g_w)^t]^{-\tau_1} \tau_2^{-\tau_1} \right)^{-\frac{1}{\tau_1}} \right) = (1+g_w)^t \, \Upsilon(\tilde{y})$$

Individual Budget Constraint: An agent's time t budget constraint is:

$$c_t + a'_{t+1} \leq w_t \varepsilon_t h_t + (1+r_t) a_t - T_t(\cdot)$$
$$c + (1+g_w) a' \leq w \varepsilon h + (1+r) a - T(\cdot)$$

where $\{c, a', a, h, w, r, \varepsilon\}$ are stationary variables. Given that the tax function $\Upsilon(\tilde{y})$ grows at rate g_w, so will the transfer function $T(h, a, \varepsilon)$ in the infinitely lived agent model. Furthermore, given that the Social Security program $\{\bar{x}, b^{ss}_i, \tau_{r,i}\}$ grows at rate g_w, so will the transfer $T(h, a, \varepsilon, x, \zeta')$ function in the life cycle model.

A.2.3 Life Cycle Model

Individual Problem: On the balanced growth path of the life cycle model, the stationary dynamic program is:

$$V_j(a, \varepsilon, x, \zeta) = \max_{c, a', h, \zeta'} \left[u(c) - \zeta' v(h) \right] + [\beta s_j (1+g_w)^{1-\sigma}] \sum_{\varepsilon'} \pi_j(\varepsilon' | \varepsilon) V_{j+1}(a', \varepsilon', x', \zeta')$$

$$\begin{aligned} \text{s.t.} \quad c + (1+g_w)a' &\leq \zeta' we(\varepsilon)h + (1+r)(a+Tr) + T(h,a,\varepsilon,x,\zeta') \\ a' &\geq \underline{a} \\ \zeta' &\in \{\mathbb{1}(j < \underline{J}_{ret}),\ \mathbb{1}(j \leq \bar{J}_{ret}) \cdot \zeta\} \end{aligned}$$

Aggregation: Aggregate consumption in the life cycle model is constructed as follows. Define the relative size of cohorts as $\mu_1 = 1$ and:

$$\mu_{j+1} = \frac{N_{t-j}}{N_t} \cdot \prod_{i=1}^{j} s_i = (1+g_n)^{-j} \prod_{i=1}^{j} s_i = \frac{s_j \mu_j}{1+g_n} \quad \forall\, j = 1, \ldots, J-1$$

Let $C_{j,t}$ be aggregate consumption per age-j agent, which is derived from the age-j agent's allocation:

$$C_{j,t} = \int (1+g_w)^t c_j(a,\varepsilon) d\lambda_j = (1+g_w)^t \int c_j(a,\varepsilon) d\lambda_j = (1+g_w)^t C_j$$

where C_j is the stationary aggregate consumption per age-j agent. Accordingly, aggregate consumption is:

$$\begin{aligned} C_t &= N_t \Big(C_{1,t} + s_1(1+g_n)^{-1} C_{2,t} + \cdots + \left(\prod_{i=1}^{J-1} s_i \right) (1+g_n)^{-(J-1)} C_{J,t} \Big) \\ &= (1+g_w)^t N_t \sum_{j=1}^{J} \mu_j C_j \\ &= (1+g)^t C \end{aligned}$$

where C is the stationary level of aggregate consumption and where we have normalized $N = 1$.

We can similarly construct the remaining aggregates $\{A, K, Y, B, G\}$ on the balanced growth path. Notably, however, labor per capita does not grow. Aggregate labor per capita is constructed as:

$$L_t = N_t \sum_{j=1}^{J} \mu_j L_j \quad \Longrightarrow \quad L = \frac{L_t}{N_t} = \sum_{j=1}^{J} \mu_j \int \varepsilon h_j(a,\varepsilon) d\lambda_j$$

which is the sum over ages of aggregate labor per age-j agent.

Stationarity: Aggregate savings per age-j agent on the balanced growth path at time t is:

$$A'_{j,t+1} = \int a'_{j,t+1}(a,\varepsilon)d\lambda_j = \int (1+g_w)^t a'_j(a,\varepsilon)d\lambda_j = (1+g_w)^t A'_j$$

and aggregate savings on the balanced growth path at time t is:

$$A'_{t+1} = N_t \sum_{j=1}^{J-1} \mu_j A'_{j,t+1} = (1+g)^{t+1} \frac{1}{1+g_n} \sum_{j=1}^{J-1} \mu_j A'_j$$

where the summation over age has an upper limit of $J-1$, which reflects that agents never save at their terminal age, J.

Likewise, aggregate wealth on the balanced growth path at time t is:

$$A_t = N_t \sum_{j=2}^{J} \mu_j \int a_{j,t} d\lambda_j = (1+g)^t \sum_{j=2}^{J} \mu_j A_j$$

where the summation over age has an lower limit of 2, which reflects that agents are endowed with zero wealth at age 1.

Accordingly, assets saved at time t must equal wealth at time $t+1$. If $A'_{t+1} = A_{t+1}$ then

$$\frac{1}{1+g_n} \sum_{j=1}^{J-1} \mu_j A'_j = \sum_{j=2}^{J} \mu_j A_j$$

which is the equilibrium condition that ensures stationarity.

[To Be Completed.]

A.3 Welfare Decomposition

Proposition 1: *If preferences are additively separable in utility over consumption, $u(c)$, and disutility over hours, $v(h)$, then welfare changes can be decomposed as:*

$$(1+\Delta_{CEV}) = (1+\Delta^{L})\underbrace{(1+\Delta^{I})(1+\Delta^{R})}_{\equiv(1+\Delta_{D})}(1+\Delta^{H})$$

Proof: Consider two economies, $i \in \{1,2\}$. Define ex ante welfare in economy $i \in \{1,2\}$ as:

$$S^i = S^i_c + S^i_h \equiv \int \mathbb{E}_0 \left[\sum_{j=1}^{J} \beta^{j-1} s_j u\left(c^i_j\right)\right] d\lambda^i_1 + \int \mathbb{E}_0 \left[\sum_{j=1}^{J} \beta^{j-1} s_j \zeta^i_{j+1} v\left(h^i_j\right)\right] d\lambda^i_1$$

Denote the Consumption Equivalent Variation (CEV) by Δ_{CEV}, which can be defined as the percent of lifetime consumption that an agent inhabiting economy $i=1$ would pay in order to inhabit economy $i=2$:

$$(1+\Delta_{CEV})^{1-\sigma} S^1_c + S^1_h = S^2$$

Furthermore, define an individual's certainty equivalent consumption as the level $\bar{c}(a,\varepsilon,x,\zeta)$ such that the individual is indifferent between consuming $\bar{c}(a,\varepsilon,x,\zeta)$ at every age with certainty and consuming according to policy function $\{c_j(a,\varepsilon,x,\zeta)\}_{j=1}^{J}$ with uncertainty. That is, $\bar{c}(a,\varepsilon,x,\zeta)$ is defined by:

$$S^i_c \equiv \int \mathbb{E}_0 \left[\sum_{j=1}^{J} \beta^{j-1} s_j u\left(c^i_j\right)\right] d\lambda^i_1 = \left(\sum_{j=1}^{J} \beta^{j-1} s_j\right) \int u\left(\bar{c}^i(a_1,\varepsilon_1,x_1,\zeta_1)\right) d\lambda^i_1$$

which implies the definition of aggregate certainty equivalent consumption:

$$\bar{C}^i \equiv \int \bar{c}^i_1(a_1,\varepsilon_1,x_1,\zeta_1) d\lambda^i_1$$

Therefore, if agents only consume their certainty equivalent consumption allocation, then they only face ex ante risk in their consumption. Define the *redistribution effect* by a comparison between consuming an individual and aggregate certainty equivalent consumption allocation:

$$\int \mathbb{E}_0 \left[\sum_{j=1}^{J} \beta^{j-1} s_j u \left((1-\omega_R^i) \bar{C}^i \right) \right] d\lambda_1^i = \left(\sum_{j=1}^{J} \beta^{j-1} s_j \right) \int u \left(\bar{c}^i(a_1, \varepsilon_1, x_1, \zeta_1) \right) d\lambda_1^i$$

which implies:

$$1-\omega_R^i = \frac{(S_c^i / \sum_{j=1}^{J} \beta^{j-1} s_j)^{\frac{1}{1-\sigma}}}{\bar{C}^i}$$

and

$$1+\Delta^R = \frac{1-\omega_R^2}{1-\omega_R^1} = \frac{(S_c^2/S_c^1)^{\frac{1}{1-\sigma}}}{\bar{C}^2/\bar{C}^1}$$

Likewise, we can define the *uncertainty effect* as a comparison between consuming at each age, the aggregate consumption allocation:

$$C^i = \sum_{j=1}^{J} \mu_j \int c_j^i(a, \varepsilon, x, \zeta) d\lambda_j^i$$

and the aggregate certainty equivalent consumption, $\bar{C}^i$. Then:

$$\int \mathbb{E}_0 \left[\sum_{j=1}^{J} \beta^{j-1} s_j u \left((1-\omega_I^i) C^i \right) \right] d\lambda_1^i = \left(\sum_{j=1}^{J} \beta^{j-1} s_j \right) \int u \left(\bar{C}^i \right) d\lambda_1^i$$

which implies:

$$1-\omega_I^i = \frac{\bar{C}^i}{C^i} \quad \text{and} \quad 1+\Delta_I = \frac{1-\omega_I^2}{1-\omega_I^1}$$

Lastly, define the labor disutility effect Δ^H as the percent of lifetime consumption that an individual would pay to change their hours allocation:

$$(1+\Delta^H)^{1-\sigma} S_c^2 = S_c^2 + (S_h^2 - S_h^1)$$

Proceeding from the definition of the CEV, we can decompose welfare as follows:

$$\begin{array}{ccccccccc}
(1+\Delta_{CEV}) & = & (1+\Delta^{L}) & \cdot & (1+\Delta^{I}) & \cdot & (1+\Delta^{R}) & \cdot & (1+\Delta^{H}) \\
\left(\dfrac{S^2 - S_h^1}{S_c^1}\right)^{\frac{1}{1-\sigma}} & = & (C^2/C^1) & \cdot & \dfrac{\bar{C}^2/\bar{C}^1}{C^2/C^1} & \cdot & \dfrac{(S_c^2/S_c^1)^{\frac{1}{1-\sigma}}}{\bar{C}^2/\bar{C}^1} & \cdot & \dfrac{\left((S^2 - S_h^1)/S_c^1\right)^{\frac{1}{1-\sigma}}}{(S_c^2/S_c^1)^{\frac{1}{1-\sigma}}}
\end{array}$$

Canceling terms on the right hand side of the expression readily shows a decomposition holds as desired. In the text, we combine $(1+\Delta^{I})(1+\Delta^{R})$ as an amalgam term, consistent with Conesa et al. (2009), to form the consumption distribution effect. Decomposing the labor hours effect follows similar reasoning. ■