Advanced Topics in Macroeconomics: Quantitative Macroeconomics

Problem Set 1 (Warm Up)

Question (1a): Write the agent's dynamic optimization problem in either sequential or recursive form.

$$v(s_0) = \max_{\{c_t, s_{t+1}\}_{t=0}^{T}} \left\{ \sum_{t=0}^{T} \beta^t \log(c_t) \quad \text{s.t.} \quad c_t + s_{t+1} \leq s_t, \quad s_0 \text{ given} \right\}$$

Question (1b): Solve for the agent's optimal paths of consumption and savings $\{c_t, s_{t+1}\}_{t=0}^{T}$ given the endowment s_0.

1. Given that $s_{T+1} = 0$ then $c_T = s_T$ from the budget constraint.
2. The Euler equation gives for all $t < T$:

$$\frac{1}{c_t} = \beta \frac{1}{c_{t+1}}$$

   which implies:

$$c_t = \beta^t c_0$$

3. Substituting $c_t = \beta^t c_0$ into each period's budget constraint gives:

$$\begin{aligned} s_1 &= s_0 - c_0 \\ s_2 &= (s_0 - c_0) - \beta c_0 \\ s_3 &= (s_0 - (1+\beta)c_0) - \beta^2 c_0 \\ &\vdots \quad \vdots \\ s_t &= s_0 - c_0 \sum_{j=0}^{t-1} \beta^j \end{aligned}$$

4. Substitute this expression for s_t from step 3 and the expression for c_t from step 2 into the terminal period budget constraint from step 1:

$$\begin{aligned} c_T &= s_T \\ \beta^T c_0 &= s_0 - c_0 \sum_{j=0}^{T-1} \beta^j \end{aligned}$$

$$c_0 = \frac{s_0}{\sum_{j=0}^{T} \beta^j}$$

Note that this implies:

$$c_t = \frac{1-\beta}{1-\beta^{T+1}} \cdot \beta^t s_0$$

5. Finally, substitute c_0 into the expression for savings in step 3:

$$\begin{aligned} s_t &= s_0 - \left(\frac{s_0}{\sum_{j=0}^{T} \beta^j} \right) \sum_{j=0}^{t-1} \beta^j \\ &= \left(1 - \frac{\sum_{j=0}^{t-1} \beta^j}{\sum_{j=0}^{T} \beta^j} \right) s_0 \end{aligned}$$

Note that this can be rewritten as:

$$s_t = \frac{1-\beta^{T+1-t}}{1-\beta^{T+1}} \cdot \beta^t s_0$$

Question (2a): Write the Bellman equation.

$$\begin{aligned} v(k) = \max_{c,k' \geq 0} \; & u(c) + \beta v(k') \\ \text{s.t.} \quad & c + k' \leq k^\alpha + (1-\delta)k \end{aligned}$$

Question (2b): Assume that $u(c) = \log(c)$ and $\delta = 1$ and solve the dynamic program in (a). To obtain a solution, guess and verify that $v(k) = A + B\log(k)$ and solve for A and B. Your answer should be an analytical expression for the value function $v(k)$, the consumption policy function $c(k)$ and the capital savings policy function $k'(k)$.

From first order conditions:

$$\frac{1}{c} = \beta v'(k')$$

Then from the envelope condition:

$$v'(k) = \alpha k^{\alpha-1} + (1-\delta)$$

Using the guess that $v(k) = A + B\log(k)$:

$$v'(k) = B \cdot \frac{1}{k}$$

This implies that:

$$\frac{1}{c} = \frac{\beta B}{k'}$$

Using the budget constraint to substitute for k':

$$c(k) = \frac{1}{1+\beta B} k^{\alpha}$$

Substitute back into the budget constraint:

$$k'(k) = \frac{\beta B}{1+\beta B} k^{\alpha}$$

Substitute $c(k)$ and $k'(k)$ into the Bellman equation and group like terms:

$$\begin{aligned} A + B\log(k) &= \log\left(\frac{1}{1+\beta B}\right) + \alpha\log(k) + \beta A + \beta B \log\left(\frac{\beta B}{1+\beta B}\right) + \alpha\beta B\log(k) \\ &= \left[\log\left(\frac{1}{1+\beta B}\right) + \beta A + \beta B\log\left(\frac{\beta B}{1+\beta B}\right)\right] + \left[\alpha + \alpha\beta B\right]\log(k) \end{aligned}$$

Then:

$$B = \alpha + \alpha\beta B$$

$$B = \frac{\alpha}{1-\alpha\beta}$$

and:

$$A = \log\left(\frac{1}{1+\beta B}\right) + \beta A + \beta B \log\left(\frac{\beta B}{1+\beta B}\right)$$

$$A = \frac{1}{1-\beta}\left(\log\left(1-\alpha\beta\right) + \frac{\alpha\beta}{1-\alpha\beta}\log\left(\alpha\beta\right)\right)$$

Lastly, substitute A and B into the value function to obtain:

$$v(k) = \frac{1}{1-\beta}\left(\log\left(1-\alpha\beta\right) + \frac{\alpha\beta}{1-\alpha\beta}\log\left(\alpha\beta\right)\right) + \frac{\alpha}{1-\alpha\beta}\log(k)$$

and into policy functions to obtain:

$$c(k) = (1-\alpha\beta)k^{\alpha}$$

$$k'(k) = \alpha\beta k^{\alpha}$$